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Some aspects of quantization of the kinetic energy of a deformed nucleus in curvilinear coordinates

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Abstract. Excited collective states of even-even nuclei are described by the Bohr Hamiltonian in curvilinear coordinates. The explicitly form of this Bohr Hamiltonians were previously obtained, but some aspects of this work remained incomplete. This paper clarifies some unfinished aspects of the quantization of the classical kinetic energy of octupole vibrations. The procedure for finding the inverse matrix is performed and the explicit form of this matrix was obtained.

Key words: quantization, curvilinear coordinates, quadrupole and octupole deformations, deformed nuclei, Bohr Hamiltonian, surface oscillations and elastic vibrations, spherical harmonics, components of moment of inertia, inverse matrix.

I. INTRODUCTION

The fundamental modes of nuclear excitation correspond to collective types of motion, such as surface oscillations and elastic vibrations. Observed nuclear levels with specific oscillation modes provide a direct explanation for the rapid increase in level density with increasing nuclear excitation [1]. In this paper, the nuclear surface is described by five expansion parameters corresponding to second-order spherical harmonics for quadrupole deformations. In [2], the total energy of quadrupole and octupole deformations was obtained, where the nuclear surface is described by seven expansion parameters corresponding to third-order spherical harmonics. But the full form of Bohr Hamiltonian with quadrupole and octupole deformations is not given. However, it should be taken into account that it is necessary to quantize the classical kinetic energy octupole vibrations of the surface of the nucleus in curvilinear coordinates.

It is known that, along with Cartesian rectangular coordinate systems, curvilinear coordinate systems can be used. In certain cases, curvilinear coordinates are more convenient than Cartesian ones. Often curvilinear coordinates are considered at the level of differential and integral calculus, in vector calculus or when transforming volume integrals. It is also important to present the parameterization of surfaces and volumes in non-Cartesian coordinate systems [3]. It should be noted that, the transition from Cartesian coordinates to curvilinear coordinates in the Schrödinger equation simplifies the problems under consideration. For example, when transitioning from Cartesian coordinates to spherical coordinates of this equation, where the spherical angular coordinates are described by Legendre polynomials, which leads to the solution of only the radial Schrödinger equation.

To describe quadrupole and octupole excitations in even-even nuclei, a Hamiltonian is used. For ease of analysis, this operator is expressed in a curvilinear coordinate system. In the work by [4], using the Pauli procedure [5], the quantization of the kinetic energy of a deformed nucleus in curvilinear coordinates for quadrupole vibrations of the nuclear surface is described in detail. In Refs. [6, 7] quantization of the classical kinetic energy associated with rotation by octupole vibrations of the nuclear surface has been performed. In this paper, the kinetic energy of a deformed nucleus with octupole vibrations of the surface is quantized for the first time and they are presented in curvilinear coordinates. As a result, the exact form of the Hamiltonian for an even-even nucleus with quadrupole and octupole deformations is obtained. This Hamiltonian is determined by seven dynamic variables: β_3, γ_3, β_3, η_3, θ_1, θ_2, θ_3.

The above-mentioned works describe in detail the quantization of classical kinetic energy associated with rotation via octupole vibrations in curvilinear coordinates. However, some aspects of this work remain incomplete. The goal of this paper is to clarify these some unfinished aspects of the quantization of classical kinetic energy of octupole vibrations.

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II. DEFINITION OF A DIRECT MATRIX

The quantum expression for kinetic energy is [4]

$$T = -\frac{\hbar^2}{2}\sum_{\mu\nu}\frac{1}{G^{1/2}}\frac{\partial}{\partial q_\mu}G^{1/2}g^{-1}_{\mu\nu}\frac{\partial}{\partial q_\nu}, \tag{1}$$

where G is the determinant of the matrix $q_{\mu\nu}$ and $q^{-1}_{\mu\nu}$ denotes the inverse matrix of $q_{\mu\nu}$. The volume element is

$$d\tau = |\sqrt{G}|dq_1.....dq_N. \tag{2}$$

To quantize kinetic energy, we use formula (1). We will write classical kinetic energy in terms of Euler angles and internal variables

$$T = \frac{1}{2}\sum_k J_k(a_{30}, a_{32})\left(\sum_i V_{kl}(\theta_1, \theta_2, \theta_3)\frac{d\theta_i}{dt}\right)^2 + E_{\rm oct}, \tag{3}$$

where $E_{\rm oct}$ – kinetic energy octupole vibrations, J_k are the components of moment of inertia, a quantity V_{kl} has the form [4]

$$V_{k,l}(\theta_1, \theta_2, \theta_3) = \begin{pmatrix} -\sin\theta_2\cos\theta_3 & \sin\theta_3 & 0 \\ \sin\theta_2\cos\theta_3 & \cos\theta_3 & 0 \\ \cos\theta_2 & 0 & 1. \end{pmatrix} \tag{4}$$

From here

$$g_{11} = B; \quad g_{22} = 2B; \qquad g_{1k} = g_{k1}; \quad \text{for} \quad k \neq 1, \qquad g_{2k} = g_{k2}; \quad \text{for} \quad k \neq 2.$$

$$g_{\mu\nu} = \sum_k J_k V_{k\mu}(\theta_i)V_{k\nu}(\theta_i), \qquad \mu, \nu \geq 3. \tag{5}$$

Explicit expression for this matrix

$$g_{\mu\nu} = \begin{pmatrix} B & 0 & 0 & 0 & 0 \\ 0 & 2B & 0 & 0 & 0 \\ 0 & 0 & (J_1\cos^2\theta_3 + J_2\sin^2\theta_3)\sin^2\theta_2 + J_3\cos^2\theta_2 & (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3 & J_3\cos\theta_2 \\ 0 & 0 & (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3 & J_1\sin^2\theta_3 + J_2\cos^2\theta_3 & 0 \\ 0 & 0 & J_3\cos\theta_2 & 0 & J_3. \end{pmatrix} \tag{6}$$

We will easily calculate the determinant of a matrix

$$G = det(g_{\mu\nu}) = 2B_3^2\sin^2\theta_2 J_1 J_2 J_3 = 32B_3^5\sin^2\theta_2 a_{32}^2\left[\left(3a_{30}^2 + 4a_{32}^2\right)^2 - 30a_{30}^2 a_{32}^2\right]. \tag{7}$$

From expression (1), it is clear that the quantum expression for the kinetic energy is determined by the inverse matrix $g^{-1}_{\mu\nu}$. Its definition requires the concepts of a transposed matrix, a matrix minor, and the cofactor of a matrix element. They are some unfinished aspects of the quantization of classical kinetic energy of octupole vibrations.

III. THE PROCEDURE FOR FINDING THE INVERSE MATRIX

We represent the inverse matrix of A as follows:

$$A^{-1} = \frac{A^T}{det(A)} \tag{8}$$

where A^T is a matrix composed of the cofactors of the corresponding elements of the transposed matrix [3].

To determine $g^{-1}_{\mu\nu}$, we rewrite the matrix (6) as follows.

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$$g_{\mu\nu} = 2B^2 \begin{pmatrix} (J_1\cos^2\theta_3 + J_2\sin^2\theta_3)\sin^2\theta_2 + J_3\cos^2\theta_2 & (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3 & J_3\cos\theta_2 \\ (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3 & J_1\sin^2\theta_3 + J_2\cos^2\theta_3 & 0 \\ J_3\cos\theta_2 & 0 & J_3. \end{pmatrix} \tag{9}$$

First, we find the matrix of minors

$$M_{ij} = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33}. \end{pmatrix} \tag{10}$$

Consider the first element of the matrix $g_{\mu\nu}$ (9) and cross out the row and column containing this element.

$$g' = \begin{pmatrix} J_1\sin^2\theta_3 + J_2\cos^2\theta_3 & 0 \\ 0 & J_3. \end{pmatrix} \tag{11}$$

This determinant, "two by two", is the minor of this element. Let's calculate:

$$D = (J_1\sin^2\theta_3 + J_2\cos^2\theta_3)\cdot J_3,$$

We write it into our matrix of minors:

$$M_{11} = (J_1\sin^2\theta_3 + J_2\cos^2\theta_3)\cdot J_3,$$

We consider the second element of the matrix $g_{\mu\nu}$ (9) and cross out the row and column containing this element.

$$g' = \begin{pmatrix} (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3 & 0 \\ J_3\cos\theta_2 & J_3. \end{pmatrix} \tag{12}$$

$$M_{12} = (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3\cdot J_3,$$

Consider the third element of the matrix $g_{\mu\nu}$ (9) and cross out the row and column containing this element

$$g' = \begin{pmatrix} (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3 & J_1\sin^2\theta_3 + J_2\cos^2\theta_3 \\ J_3\cos\theta_2 & 0. \end{pmatrix} \tag{13}$$

$$M_{13} = -J_3\cos\theta_2\cdot(J_1\sin^2\theta_3 + J_2\cos^2\theta_3),$$

Consider the fourth element of the matrix $g_{\mu\nu}$ (9) and cross out the row and column containing this element

$$g' = \begin{pmatrix} (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3 & J_3\cos\theta_2 \\ 0 & J_3. \end{pmatrix} \tag{14}$$

$$M_{21} = (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3\cdot J_3,$$

Consider the fifth element of the matrix $g_{\mu\nu}$ (9) and cross out the row and column containing this element

$$g' = \begin{pmatrix} (J_1\cos^2\theta_3 + J_2\sin^2\theta_3)\sin^2\theta_2 + J_3\cos^2\theta_2 & J_3\cos\theta_2 \\ J_3\cos\theta_2 & J_3. \end{pmatrix} \tag{15}$$

$$M_{22} = [(J_1\cos^2\theta_3 + J_2\sin^2\theta_3)\sin^2\theta_2 + J_3\cos^2\theta_2]\cdot J_3 - J_3^2\cos^2\theta_2 =$$

$$= (J_1\cos^2\theta_3 + J_2\sin^2\theta_3)\sin^2\theta_2\cdot J_3 + J_3^2\cos^2\theta_2 - J_3^2\cos^2\theta_2 = (J_1\cos^2\theta_3 + J_2\sin^2\theta_3)\sin^2\theta_2\cdot J_3,$$

Consider the sixth element of the matrix $g_{\mu\nu}$ (9) and cross out the row and column containing this element.

$$g' = \begin{pmatrix} (J_1\cos^2\theta_3 + J_2\sin^2\theta_3)\sin^2\theta_2 + J_3\cos^2\theta_2 & (J_2 - J_1)\sin\theta_2\cos\theta_3\sin\theta_3 \\ J_3\cos\theta_2 & 0. \end{pmatrix} \tag{16}$$

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$$M_{23} = -J_3 \cos\theta_2 (J_2 - J_1) \sin\theta_2 \cos\theta_3 \sin\theta_3$$

Consider the seventh element of the matrix $g_{\mu\nu}$ (9) and cross out the row and column, in which the given element is located

$$g' = \begin{pmatrix} (J_2 - J_1)\sin\theta_2 \cos\theta_3 \sin\theta_3 & J_3 \cos\theta_2 \\ J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3 & 0. \end{pmatrix} \tag{17}$$

$$M_{31} = -J_3 \cos\theta_2 \cdot (J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3)$$

Consider the eighth element of the matrix $g_{\mu\nu}$ (9) and cross out the row and column in which the given element is located

$$g' = \begin{pmatrix} (J_1 \cos^2\theta_3 + J_2 \sin^2\theta_3)\sin^2\theta_2 + J_3 \cos^2\theta_2 & J_3 \cos\theta_2 \\ (J_2 - J_1)\sin\theta_2 \cos\theta_3 \sin\theta_3 & 0 \end{pmatrix} \tag{18}$$

$$M_{32} = -J_3 \cos\theta_2 \cdot (J_2 - J_1) \sin\theta_2 \cos\theta_3 \sin\theta_3$$

Consider the ninth element of the matrix $g_{\mu\nu}$ (9) and We cross out the row and column containing the given element.

$$g' = \begin{pmatrix} (J_1 \cos^2\theta_3 + J_2 \sin^2\theta_3)\sin^2\theta_2 + J_3 \cos^2\theta_2 & (J_2 - J_1)\sin\theta_2 \cos\theta_3 \sin\theta_3 \\ (J_2 - J_1)\sin\theta_2 \cos\theta_3 \sin\theta_3 & J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3. \end{pmatrix} \tag{19}$$

$$M_{33} = [(J_1 \cos^2\theta_3 + J_2 \sin^2\theta_3)\sin^2\theta_2 + J_3 \cos^2\theta_2] \cdot [J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3] - (J_2 - J_1)^2 \sin^2\theta_2 \cos^2\theta_3 \sin^2\theta_3$$

$$M_{ij} = 2B^2 \cdot$$

$$\cdot \begin{pmatrix} (J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3) \cdot J_3 & (J_2 - J_1)\sin\theta_2 & -J_3 \cos\theta_2 \cdot \\ & \cos\theta_3 \sin\theta_3 \cdot J_3 & (J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3) \\ (J_2 - J_1)\sin\theta_2 \cdot & (J_1 \cos^2\theta_3 + J_2 \sin^2\theta_3) & -J_3 \cos\theta_2 (J_2 - J_1) \\ \cos\theta_3 \sin\theta_3 J_3 & \sin^2\theta_2 \cdot J_3 & \sin\theta_2 \cos\theta_3 \sin\theta_3 \\ -J_3 \cos\theta_2 \cdot & -J_3 \cos\theta_2 \cdot (J_2 - J_1) & [(J_1 \cos^2\theta_3 + J_2 \sin^2\theta_3)\sin^2\theta_2 + J_3 \cos^2\theta_2] \cdot \\ (J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3) & \sin\theta_2 \cos\theta_3 \sin\theta_3 & [J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3] - \\ & & -(J_2 - J_1)^2 \sin^2\theta_2 \cos^2\theta_3 \sin^2\theta_3. \end{pmatrix} \tag{20}$$

In the matrix of minors (20) it is necessary to change the signs of the following elements: M_{12}, M_{21}, M_{23}, M_{32}.

$$M_{ij} = 2B^2 \cdot$$

$$\cdot \begin{pmatrix} (J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3) \cdot J_3 & -(J_2 - J_1)\sin\theta_2 & -J_3 \cos\theta_2 \cdot \\ & \cos\theta_3 \sin\theta_3 \cdot J_3 & (J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3) \\ -(J_2 - J_1)\sin\theta_2 \cdot & (J_1 \cos^2\theta_3 + J_2 \sin^2\theta_3) & J_3 \cos\theta_2 (J_2 - J_1) \\ 4 \quad \cos\theta_3 \sin\theta_3 J_3 & \sin^2\theta_2 \cdot J_3 & \sin\theta_2 \cos\theta_3 \sin\theta_3 \\ -J_3 \cos\theta_2 \cdot & J_3 \cos\theta_2 \cdot (J_2 - J_1) & [(J_1 \cos^2\theta_3 + J_2 \sin^2\theta_3)\sin^2\theta_2 + J_3 \cos^2\theta_2] \cdot \\ (J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3) & \sin\theta_2 \cos\theta_3 \sin\theta_3 & [J_1 \sin^2\theta_3 + J_2 \cos^2\theta_3] - \\ & & -(J_2 - J_1)^2 \sin^2\theta_2 \cos^2\theta_3 \sin^2\theta_3. \end{pmatrix} \tag{21}$$

(21) matrix of algebraic complements of the corresponding elements.

We find the transposed matrix of algebraic complements M_{ij}^T.

$$M_{ij}^T = 2B^2 \cdot$$

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$$\cdot \begin{pmatrix} (J_1\sin^2\theta_3 + J_2\cos^2\theta_3)\cdot J_3 & -(J_2-J_1)\sin\theta_2 & -J_3\cos\theta_2\cdot \\ & \cos\theta_3\sin\theta_3\cdot J_3 & (J_1\sin^2\theta_3 + J_2\cos^2\theta_3) \\ -(J_2-J_1)\sin\theta_2\cdot & (J_1\cos^2\theta_3 + J_2\sin^2\theta_3) & J_3\cos\theta_2(J_2-J_1) \\ \cos\theta_3\sin\theta_3 J_3 & \sin^2\theta_2\cdot J_3 & \sin\theta_2\cos\theta_3\sin\theta_3 \\ & & [(J_1\cos^2\theta_3 + J_2\sin^2\theta_3)\sin^2\theta_2 + J_3\cos^2\theta_2]\cdot \\ -J_3\cos\theta_2\cdot & J_3\cos\theta_2\cdot(J_2-J_1) & [J_1\sin^2\theta_3 + J_2\cos^2\theta_3]- \\ (J_1\sin^2\theta_3 + J_2\cos^2\theta_3) & \sin\theta_2\cos\theta_3\sin\theta_3 & -(J_2-J_1)^2\sin^2\theta_2\cos^2\theta_3\sin^2\theta_3. \end{pmatrix} \tag{22}$$

(22) is the transposed matrix of algebraic complements. Thus we obtain the inverse matrix.

$$g^{-1}_{\mu\nu} = \frac{M^T_{\mu\nu}}{|G|}. \tag{23}$$

The following should be noted here:

$$g^{-1}_{55} = \sin^2\theta_2[J_1J_2\cos^4\theta_3 + J_2\cos^2\theta_3(J_3\cot^2\theta_2 + 2J_1\sin^2\theta_3) + J_1\sin^2\theta_3(J_3\cot^2\theta_2 + J_2\sin^2\theta_3)].$$

This expression differs from same the expression in [4].

It is known that the total energy of octupole deformation has the form:

$$T_{\beta_3} = \frac{B_3}{2}(\dot{a}^2_{30} + 2\dot{a}^2_{32}) + \sum_{\lambda=1}^{3}\frac{\hbar^2\hat{I}^2_\lambda}{J^{(3)}_\lambda}, \tag{24}$$

here $J^{(3)}_\lambda$ are the projections of the total moment of inertia for the octupole deformations:

$$J^{(3)}_1 = 4B_3\left(3a^2_{30} + 4a^2_{32} + \sqrt{30}a_{30}a_{32}\right), \quad J^{(3)}_2 = 4B_3\left(3a^2_{30} + 4a^2_{32} - \sqrt{30}a_{30}a_{32}\right), \quad J^{(3)}_3 = 8B_3a^2_{32}, \tag{25}$$

here

$$a'_{30} = \beta_3\cos\eta, \qquad a'_{32} = a'_{3,-2} = \frac{\beta_3\sin\eta}{\sqrt{2}}, \tag{26}$$

Repeating the above procedures, we obtain operator of kinetic energy of octupole deformation in curvilinear coordinates [6, 7]:

$$\hat{T}_{\beta_3,\eta} = -\frac{\hbar^2}{2B_3}\left[\frac{1}{\beta^4_3}\frac{\partial}{\partial\beta_3}\left(\beta^4_2\frac{\partial}{\partial\beta_3}\right) + \frac{1}{\beta^2_3}\frac{\partial^2}{\partial\eta^2} + \frac{24\cos^2 2\eta - 6\cos 2\eta}{5 + 5\cos 2\eta + 8\cos^2 2\eta}\frac{\cos\eta}{\beta^2_3\sin\eta}\frac{\partial}{\beta^2_3\partial\eta}\right]. \tag{27}$$

IV. KINETIC ENERGY OPERATORS OF EVEN-EVEN NUCLEI WITH QUADRUPOLE AND OCTUPOLE DEFORMATIONS

We compare the resulting expression (27) with the operator of kinetic energy of quadrupole deformation:

$$\hat{T}_{\beta_2,\gamma} = -\frac{\hbar^2}{2B_2}\left[\frac{1}{\beta^4_2}\frac{\partial}{\partial\beta_2}\left(\beta^4_2\frac{\partial}{\partial\beta_2}\right) + \frac{1}{\beta^2_2}\frac{1}{\sin 3\gamma}\frac{\partial}{\partial\gamma}\left(\sin 3\gamma\frac{\partial}{\partial\gamma}\right)\right], \tag{28}$$

Then total kinetic energy operator of even-even nuclei with quadrupole and octupole deformations has the form:

$$\hat{T}_{\beta_2,\beta_3,\eta,\gamma} = \hat{T}_{\beta_2,\gamma} + \hat{T}_{\beta_3,\eta}. \tag{29}$$

And the total Bohr Hamiltonian is expressed as

$$\hat{H} = \hat{T}_{\beta_2,\beta_3,\eta,\gamma} + \hat{T}_{rot} + V(\beta_2,\beta_3,\eta,\gamma), \tag{30}$$

where

$$\hat{T}_{\rm rot} = \sum_{\kappa}^{3}\frac{\hbar^2\hat{I}^2_\kappa}{J^2_\kappa}, \tag{31}$$

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the operator of rotational energy, where I_κ(κ=1,2,3)–components total angular momentum, $J\kappa$-components the moments of inertia tensor even-even nuclei with quadrupole and octupole deformations [7].

The general solution of the Schrödinger equation with the operator (30) depends on the chosen potential energy $V(\beta_2, \beta_3, \gamma, \eta)$. Usually this problem can be solved by choosing the phenomenological form of this potential energy. Often the general form of potential energy is expressed in following:

$$V(\beta_2, \beta_3, \gamma, \eta) = V(\beta_2) + V(\beta_3) + \frac{V(\gamma)}{\beta_2^2} + \frac{V(\eta)}{\beta_3^2}, \tag{32}$$

where V(β_2), V(β_3), V(γ) and V(η) – potential energies of β_2-, β_3-, γ- and η-vibrations, separately.

A complete numerical solution to the Schrödinger equation with the operator (30) has not yet been found, and therefore, in all models of the atomic nucleus, the study of collective excitations is carried out by introducing simplifying assumptions. However, here we will consider how these simplifications affect the kinetic energy operators of deformations.

Next, we assume that the variable η (γ) takes a fixed value, i.e., η=η_{eff} (γ=γ_{eff}) [8–11]:

$$\hat{T}_{\beta_2,\beta_3} = -\frac{\hbar^2}{2B_2}\frac{1}{\beta_2^3}\frac{\partial}{\partial\beta_2}\left(\beta_2^3\frac{\partial}{\partial\beta_2}\right) - \frac{\hbar^2}{2B_3}\frac{1}{\beta_3^3}\frac{\partial}{\partial\beta_3}\left(\beta_3^3\frac{\partial}{\partial\beta_3}\right). \tag{33}$$

This expression describes kinetic energy operator even-even nuclei with effective triaxiality.

Another assumption is considered, where η=γ=0 [12–17]:

$$\hat{T}_{\beta_2,\beta_3} = -\frac{\hbar^2}{2B_2}\frac{1}{\beta_2^2}\frac{\partial}{\partial\beta_2}\left(\beta_2^2\frac{\partial}{\partial\beta_2}\right) - \frac{\hbar^2}{2B_3}\frac{1}{\beta_3^2}\frac{\partial}{\partial\beta_3}\left(\beta_3^2\frac{\partial}{\partial\beta_3}\right), \tag{34}$$

or a simpler version [18, 19]:

$$\hat{T}_{\beta_2,\beta_3} = -\frac{\hbar^2}{2B_2}\frac{\partial^2}{\partial\beta_2^2} - \frac{\hbar^2}{2B_3}\frac{\partial^2}{\partial\beta_3^2}. \tag{35}$$

And a some works use following versions [20, 21]:

$$\hat{T}_{\beta_2,\beta_3} = -\frac{\hbar^2}{2B_2}\frac{1}{\beta_3^2}\frac{\partial}{\partial\beta_2}\left(\beta_3^2\frac{\partial}{\partial\beta_2}\right) - \frac{\hbar^2}{2B_3}\frac{1}{\beta_3^3}\frac{\partial}{\partial\beta_3}\left(\beta_3^3\frac{\partial}{\partial\beta_3}\right). \tag{36}$$

Expressions (34), (35), (36) describe the kinetic energy operator of axially symmetric even-even nuclei.

Conclusion

The resulting collective Bohr Hamiltonian generalizes the coupled quadrupole and octupole oscillations. This Hamiltonian describe low-energy states with alternating-parity of the collective motion in nuclei. Some aspects of obtaining an explicit form for the Bohr Hamiltonian with quadrupole and octupole deformations are clarified. Specifically, the procedure for finding the inverse matrix for the kinetic energy of an octupole deformation in curvilinear coordinates is considered. Various simplifying approximations for solving the Schrödinger equation with the Bohr Hamiltonian are presented for tri-axially and axially-symmetric deformed nuclei.

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