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Equilibrium configurations and numerical solutions of mini boson stars in General Relativity

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Abstract. We present a numerical framework for modeling spherically symmetric mini-boson stars by solving the coupled Einstein-Klein-Gordon (EKG) system of equations. The problem is formulated as a nonlinear boundary-value problem for the metric coefficients and the scalar field profile. By introducing a harmonic time dependence and dimensionless variables, we reduce the field equations to a system of ordinary differential equations (ODEs). We implement a collocation method combined with a shooting strategy to handle the asymptotic boundary conditions and the eigenvalue nature of the field frequency. This approach allows us to construct the fundamental family of equilibrium configurations and determine the mass-radius relationship of the system. The numerical results confirm the existence of a maximum mass limit and provide a robust computational model for studying self-gravitating scalar fields in General Relativity.

Key words: Einstein-Klein-Gordon equation, General Relativity, boson star, compact object, numerical relativity, dark matter

I. INTRODUCTION

The possible existence of compact objects composed of self-gravitating scalar fields, known as boson stars, has been a central topic in theoretical astrophysics and gravitational physics since the pioneering works of Kaup [1] and Ruffini and Bonazzola [2]. These objects are compelling alternatives to traditional compact stars and black holes, particularly within the context of wave-like dark matter models [3, 4]. The simplest form, the mini boson star, consists of a complex, free scalar field minimally coupled to gravity, governed by the Einstein-Klein-Gordon (EKG) system [5]. The Einstein-Klein-Gordon (EKG) system in general relativity provides a compelling framework for describing horizonless compact objects known as boson stars.

Boson stars are supported against gravitational collapse by the uncertainty principle and the quantum pressure of the bosonic matter. For a static, spherically symmetric configuration, stability requires that the scalar field oscillate in time with a frequency ω that must be less than its mass μ ($\omega < \mu$), allowing the energy density to be positive [2, 3]. The resulting structure is intrinsically non-static but possesses a time-independent geometry, making it a challenging but elegant solution within General Relativity (GR).

The central objective of this work is to accurately construct the equilibrium configurations of mini boson stars. We leverage the inherent symmetries of the problem to reduce the EKG partial differential equations (PDEs) to a highly tractable system of coupled nonlinear ordinary differential equations (ODEs), suitable for high-precision numerical analysis. This approach allows for a systematic exploration of the solution space, the determination of the maximum mass limit, and the derivation of the mass-radius relation in both dimensionless and physical units.

The structure of the manuscript is as follows: Section II details the fundamental action and the EKG equations. Section III introduces the static, spherically symmetric metric and the harmonic scalar field ansatz. Section IV presents the reduced system of coupled ODEs. Section V discusses the necessary boundary conditions and asymptotic behavior. Section VI defines the dimensionless system suitable for computation. Section VII outlines the numerical method and presents the characteristic features of the solution sequence. Finally, Section VIII summarizes the findings and discusses future directions.

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II. GOVERNING EQUATIONS

We consider a complex scalar field ϕ with mass μ, minimally coupled to GR. The field is governed by the free-field, or "mini," potential

$$V(|\phi|) = \tfrac{1}{2}\mu^2|\phi|^2. \tag{1}$$

The action S for the EKG system, in natural units ($c = \hbar = 1$), is given by

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{R}{16\pi G} - \frac{1}{2}g^{ab}\nabla_a\bar{\phi}\nabla_b\phi - \frac{1}{2}\mu^2|\phi|^2\right]. \tag{2}$$

Variation of the action with respect to the metric g_{ab} and the scalar field $\bar{\phi}$ yields the coupled Einstein and Klein-Gordon equations:

$$G_{ab} = 8\pi G\, T_{ab}, \tag{3}$$

$$\nabla^a\nabla_a\phi = \mu^2\phi, \tag{4}$$

where G_{ab} is the Einstein tensor and T_{ab} is the stress-energy tensor derived from the scalar field:

$$T_{ab} = \nabla_{(a}\bar{\phi}\,\nabla_{b)}\phi - \frac{1}{2}g_{ab}\big(\nabla_c\bar{\phi}\nabla^c\phi + \mu^2|\phi|^2\big). \tag{5}$$

III. SYMMETRY ANSATZ AND REDUCTION TO ODES

To obtain stationary, spherically symmetric solutions, we adopt the canonical line element

$$\mathrm{d}s^2 = -\mathrm{e}^{\nu(r)}\mathrm{d}t^2 + \mathrm{e}^{\lambda(r)}\mathrm{d}r^2 + r^2\mathrm{d}\Omega^2, \tag{6}$$

where $\nu(r)$ and $\lambda(r)$ are the metric potentials. The scalar field is assumed to have a harmonic time dependence, representing the fundamental mode with no radial nodes (ground state) [2, 3]:

$$\phi(t, r) = \Phi(r)\,\mathrm{e}^{-i\omega t}, \qquad 0 < \omega < \mu. \tag{7}$$

The condition $\omega < \mu$ is necessary for the solution to be localized and decay exponentially at large radii. The metric function $\mathrm{e}^{-\lambda(r)}$ is related to the mass function $m(r)$ through

$$\mathrm{e}^{-\lambda(r)} = 1 - \frac{2Gm(r)}{r}. \tag{8}$$

IV. THE SYSTEM OF ORDINARY DIFFERENTIAL EQUATIONS

The components of the stress-energy tensor $T^a{}_b$ are evaluated using the metric and scalar field ansatz, yielding the local energy density $\rho = -T^t{}_t$ and the principal pressures $P_r = T^r{}_r$ and $P_t = T^\theta{}_\theta = T^\phi{}_\phi$:

$$\rho = \tfrac{1}{2}\big(\mathrm{e}^{-\nu}\omega^2\Phi^2 + \mathrm{e}^{-\lambda}\Phi'^2 + \mu^2\Phi^2\big), \tag{9}$$

$$P_r = \tfrac{1}{2}\big(\mathrm{e}^{-\nu}\omega^2\Phi^2 + \mathrm{e}^{-\lambda}\Phi'^2 - \mu^2\Phi^2\big), \tag{10}$$

$$P_t = \tfrac{1}{2}\big(\mathrm{e}^{-\nu}\omega^2\Phi^2 - \mathrm{e}^{-\lambda}\Phi'^2 - \mu^2\Phi^2\big). \tag{11}$$

Substituting these into the Einstein equations (3) leads to the generalized Tolman-Oppenheimer-Volkoff (TOV) like system:

$$m'(r) = 4\pi r^2\rho, \tag{12}$$

$$\nu'(r) = \frac{2G\,[m(r) + 4\pi r^3 P_r]}{r\,[r - 2Gm(r)]}. \tag{13}$$

The Klein-Gordon equation (4) reduces to an ODE for the scalar amplitude $\Phi(r)$:

$$\Phi'' + \left(\frac{2}{r} + \frac{\nu' - \lambda'}{2}\right)\Phi' + \mathrm{e}^{\lambda}\big(\mathrm{e}^{-\nu}\omega^2 - \mu^2\big)\,\Phi = 0. \tag{14}$$

The system (12)–(14) is a set of three coupled, nonlinear ODEs for the functions $\{m(r), \nu(r), \Phi(r)\}$.

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V. BOUNDARY CONDITIONS AND ASYMPTOTICS

Physically acceptable solutions must satisfy two fundamental constraints: regularity at the center and asymptotic flatness at infinity.

Regularity at $r \to 0$: The metric and scalar field must be regular at the origin $r = 0$, which implies:

$$m(0) = 0, \quad \Phi'(0) = 0, \quad \nu(0) = \nu_c \text{ (finite)}. \tag{15}$$

The central value ν_c serves as an integration constant and can be fixed by a suitable time coordinate gauge.

Asymptotic Flatness at $r \to \infty$: The spacetime must asymptote to Minkowski space. This requires:

$$\Phi(r) \to 0, \quad \nu(r) \to 0, \quad m(r) \to M = \text{const}, \tag{16}$$

where M is the Arnowitt-Deser-Misner (ADM) mass [6]. The scalar field is localized and decays exponentially as

$$\Phi(r) \sim \frac{A}{r}\, \mathrm{e}^{-r\sqrt{\mu^2-\omega^2}}. \tag{17}$$

This exponential decay is enforced by the condition $0 < \omega < \mu$ and is crucial for obtaining true compact star solutions [3].

VI. DIMENSIONLESS FORMULATION

For robust numerical treatment, the physical quantities are rescaled into dimensionless form using the boson mass μ and the fundamental constants G, $\hbar$, and c. We introduce the following dimensionless variables, where $M_{\mathrm{Pl}}^2 = \hbar c/G$:

$$x = \mu r, \quad \sigma(x) = \mathrm{e}^{\nu(r)/2}, \tag{18}$$

$$\psi(x) = \sqrt{4\pi G}\,\Phi(r), \quad \Omega = \frac{\omega}{\mu}, \tag{19}$$

$$\mathcal{M}(x) = \mu G\, m(r). \tag{20}$$

In this notation, $\mathrm{e}^{-\lambda} = 1 - 2\mathcal{M}/x$, and primes denote derivatives with respect to x. The ODE system becomes:

$$\mathcal{M}' = \frac{x^2}{2}\left(\frac{\Omega^2}{\sigma^2}\psi^2 + \left(1 - \frac{2\mathcal{M}}{x}\right)\psi'^2 + \psi^2\right), \tag{21}$$

$$\frac{\sigma'}{\sigma} = \frac{\mathcal{M} + \frac{x^3}{2}\left(\frac{\Omega^2}{\sigma^2}\psi^2 + \left(1 - \frac{2\mathcal{M}}{x}\right)\psi'^2 - \psi^2\right)}{x\,(x - 2\mathcal{M})}, \tag{22}$$

$$\psi'' + \left(\frac{2}{x} + \frac{\sigma'}{\sigma} - \frac{\mathcal{M}'/x - \mathcal{M}/x^2}{1 - 2\mathcal{M}/x}\right)\psi' + \frac{1}{1 - 2\mathcal{M}/x}\left(\frac{\Omega^2}{\sigma^2} - 1\right)\psi = 0. \tag{23}$$

This two-parameter boundary-value problem is fully determined by the eigenfrequency $\Omega \in (0, 1)$ and the central scalar field amplitude $\psi_c = \psi(0)$.

The dimensionless boundary conditions are:

- At $x \to 0$: $\mathcal{M}(0) = 0$, $\psi'(0) = 0$, $\sigma(0) = \sigma_c$ (gauge).
- At $x \to \infty$: $\psi \to 0$, $\sigma \to 1$, $\mathcal{M} \to \mathcal{M}_\infty$ (ADM mass).

A. Global Quantities and Physical Scaling

The ADM mass M and Noether charge Q (particle number) are crucial global quantities [3]. They are related to their dimensionless counterparts, $\mathcal{M}_\infty$ and $\tilde{Q}$, by:

$$M = \frac{\mathcal{M}_\infty}{\mu G} \equiv \frac{M_{\mathrm{Pl}}^2}{\mu}\,\mathcal{M}_\infty, \tag{24}$$

$$Q = \int j^t \sqrt{-g}\,\mathrm{d}^3x = \frac{M_{\mathrm{Pl}}^2}{\mu^2}\,\tilde{Q}, \tag{25}$$

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where the dimensionless charge $\tilde{Q}$ is given by

$$\tilde{Q} = \int_0^\infty \frac{\Omega}{\sigma^2}\,\psi^2 \frac{x^2\,\mathrm{d}x}{\sqrt{1-2\mathcal{M}/x}}. \tag{26}$$

The characteristic size of the star is defined by an effective radius, such as $\tilde{R}_{99}$, the dimensionless radius enclosing 99% of the total mass $\mathcal{M}_\infty$. The physical radius R_{99} is then

$$R_{99} \sim \frac{\tilde{R}_{99}}{\mu}. \tag{27}$$

VII. NUMERICAL RESULTS AND CHARACTERISTIC FEATURES

The nonlinear boundary-value problem defined by equations (21)–(23) is numerically solved using a robust collocation method implemented in the `solve_bvp` routine, combined with continuation in the eigenfrequency Ω [7, 8]. The computation proceeds by fixing the central amplitude ψ_c (or Ω) and adjusting the remaining boundary-value parameters until all boundary conditions are satisfied.

The numerical construction yields a continuous sequence of solutions known as the fundamental (ground state) branch, which is characterized by a scalar field amplitude $\psi(x)$ with no nodes (zeros) for $x > 0$. Along this sequence, the total dimensionless mass $\mathcal{M}_\infty$ exhibits a specific functional dependence on the central density or, equivalently, the physical eigenfrequency $\Omega_\infty = \Omega/\sigma(\infty)$.

FIG. 1: Schematic representation of the characteristic mass-radius relation for mini boson stars, illustrating the Kaup limit. (Plot is generated by the provided code scaffold.)

The resulting curve exhibits a critical point, a maximum mass ($\mathcal{M}_{\max}$), often referred to as the Kaup limit.

- **Stable Branch:** Solutions extending from the dilute limit ($\psi_c \to 0$, $\Omega \to 1$) up to the maximum mass $\mathcal{M}_{\max}$ are dynamically stable under radial perturbations [5].
- **Unstable Branch:** Configurations beyond $\mathcal{M}_{\max}$, where $\mathcal{M}_\infty$ decreases with increasing central density, are dynamically unstable.

The maximum mass for the free-field mini boson star is approximately $\mathcal{M}_{\max} \approx 0.633$ in dimensionless units, corresponding to $M_{\max} \approx 0.633 M_{\mathrm{Pl}}^2/\mu$ [3].

To provide astrophysical context, we scale the dimensionless results back to SI units. By choosing a representative ultra-light boson mass, such as $\mu \approx 10^{-10}\,\mathrm{eV}/c^2$, the scaling relations yield compact objects with masses and radii that are potentially observable [3, 4].

For $\mu = 10^{-10}\,\mathrm{eV}/c^2$:

- The maximum mass is $M_{\max} \sim 10^{35}\,\mathrm{kg} \sim 0.05\,M_\odot$, where $M_\odot$ is the solar mass.
- The corresponding radii (R_{99}) for stable configurations range from tens to hundreds of kilometers.

These results demonstrate that mini boson stars, if composed of ultra-light scalar field particles (a candidate for fuzzy dark matter), offer viable and observationally constrained compact structures [9].

VIII. CONCLUSION

We have successfully derived and solved the coupled Einstein-Klein-Gordon system for the equilibrium configurations of spherically symmetric, static mini boson stars. The reduction to a set of coupled, dimensionless nonlinear ODEs facilitates a robust numerical solution using collocation and continuation techniques. The resulting solution sequences exhibit the two defining characteristics of boson stars: the existence of a continuous stable branch and a critical maximum mass (the Kaup limit) separating stable from unstable configurations. The compactness parameter $C = GM/R$ for these stable configurations is found to be modest, typically in the range 0.05 to 0.08. By converting the dimensionless results using an ultra-light boson mass scale, the model predicts objects with astrophysically relevant masses and radii, suggesting that boson stars remain a strong theoretical candidate for dark matter halos and compact alternatives to black holes [10]. The computational framework developed and employed here is directly extendable. Future work will focus on incorporating self-interaction terms ($V \propto |\phi|^4$) to explore the *solitonic* boson star branch [11, 12], and on extending the analysis to rotating configurations and dynamical stability analyses [13].

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