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Dynamics of test particles around Schwarzschild-like black holes

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Abstract. In this study, the motion of neutral test particles around a spherically symmetric Schwarzschild-like black hole is investigated. The black hole solution is described by a metric includes special parameter a/M which imitates the mass of the black hole. The effective potential for test particles in this spacetime is derived, revealing its radial dependence. The equations of motions that corresponds to the type of the black hole geometry were derived. The trajectory of the test particle is found and explained the effect of special parameter on it. Calculations were done for equatorial and non-equatorial cases their differences were shown at different parameter values.

Keywords: black hole, geodesic motion, trajectory of motion, Schwarzschild-like black hole, equations of motion, particle motion.

I. INTRODUCTION

Astrophysics came to encompass black holes as one of the main teaching realms of theory and observation. These exotic astrophysical objects do not just constitute solutions of Einstein's equations of gravitation but also act as natural laboratories for probing the predictions of general relativity with a strong-field dose. The past few decades have witnessed remarkable progress in observational astronomy, and specifically, the breakthrough observations by the Event Horizon Telescope collaboration [1] have fueled the resurgence of interest in the theoretical studies of black hole spacetimes and their astrophysical implications. There has also been a shift away from treating black holes merely as mathematical solutions, looking at them now as physical entities whose property can be probed and constrained through interactions with matter and radiation around them.

Studying the motion of test particles and light rays around black holes is one of the effective ways of learning about the nature of black holes. Due to some important astrophysical applications such as accretion of matter, ejection of relativistic jets, and production of gravitational waves, the dynamics of massive and massless particles around black holes gets highly involved. Besides, knowledge about spacetime geometry and how any modification of general relativity could affect such geometry at astrophysical scales will gain from the analysis of the stability of orbits and trajectories of infalling particles [2, 3]. Thus, considerable attention has been paid to establishing and analyzing the geodesic equations in different black hole backgrounds, eventually forming solid analytical grounds and enormous literature on this subject [4–9].

In this work, we focus on the study of trajectories of test particles in Schwarzschild-like spacetimes. By examining the equations of motion, we aim to determine how additional parameters introduced in generalized Schwarzschild metrics affect the dynamics of particles. The stability of circular orbits, the onset of plunging trajectories, and the overall qualitative behavior of geodesics depend sensitively on these parameters, making their investigation an important step for both theoretical and phenomenological considerations. Since the geodesic equations in such backgrounds are typically nonlinear and analytically intractable, we employ numerical methods to solve them with high precision. This approach allows us to efficiently explore the parameter space and to visualize the effect of the metric's deformation parameters on the particle orbits.

Throughout this paper, we adopt the metric signature $(-,+,+,+)$ and natural units where the gravitational constant G and the speed of light c are set to unity. Greek indices will be used to denote spacetime coordinates $(0,1,2,3)$, while Latin indices will denote spatial components $(1,2,3)$. Our results are presented with an emphasis on the physical interpretation of trajectories and the influence of the additional metric parameter on the motion of test particles.

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II. PARTICLE TRAJECTORY AROUND SCHWARZSCHILD-LIKE BH

The line element describing the geometry of non-rotating spherically symmetric Schwarzschild-like BH is given by [10]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{1}$$

where

$$f(r) = 1 - \frac{2Me^{-a/r}}{r}. \tag{2}$$

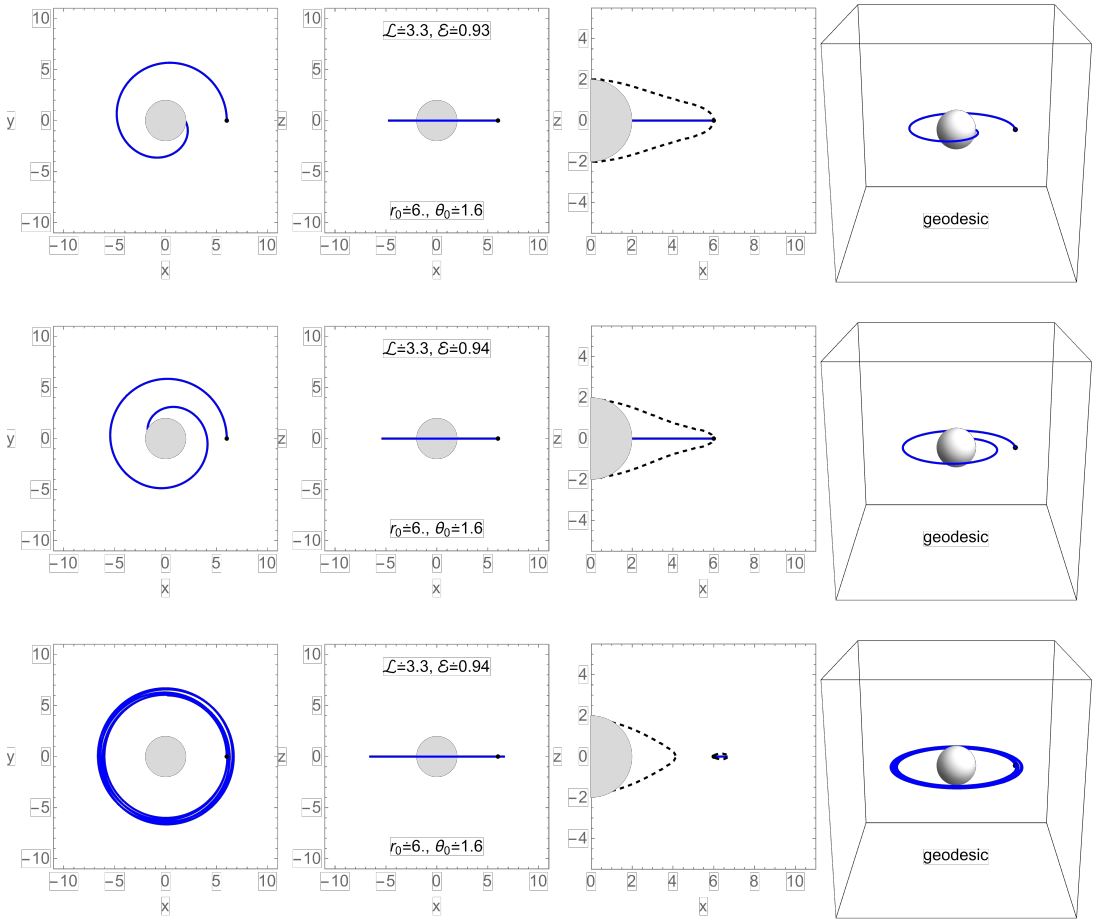


FIG. 1. Dependence of trajectory on parameter a/M in equatorial case. The values of the angular momentum, energy, initial orbit are fixed.

Also, when $a/M = 0$, the metric (1) expresses the pure Schwarzschild spacetime.The event horizon of the black hole is positioned at the real root where $g^{rr} = 0$, so parameter takes values from $(0, 0.7)$. According to general relativity, we write the Lagrange function of the particle in the form [11, 12]:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x^\mu} \dot{x^\nu}. \tag{3}$$

Also considering the normalization condition:

$$g_{\mu\nu} \dot{x^\mu} \dot{x^\nu} = -f(r)\dot{t^2} + \frac{\dot{r^2}}{f(r)} + r^2\dot{\theta^2} + r^2 \sin^2\theta \dot{\phi^2} = -\kappa, \tag{4}$$

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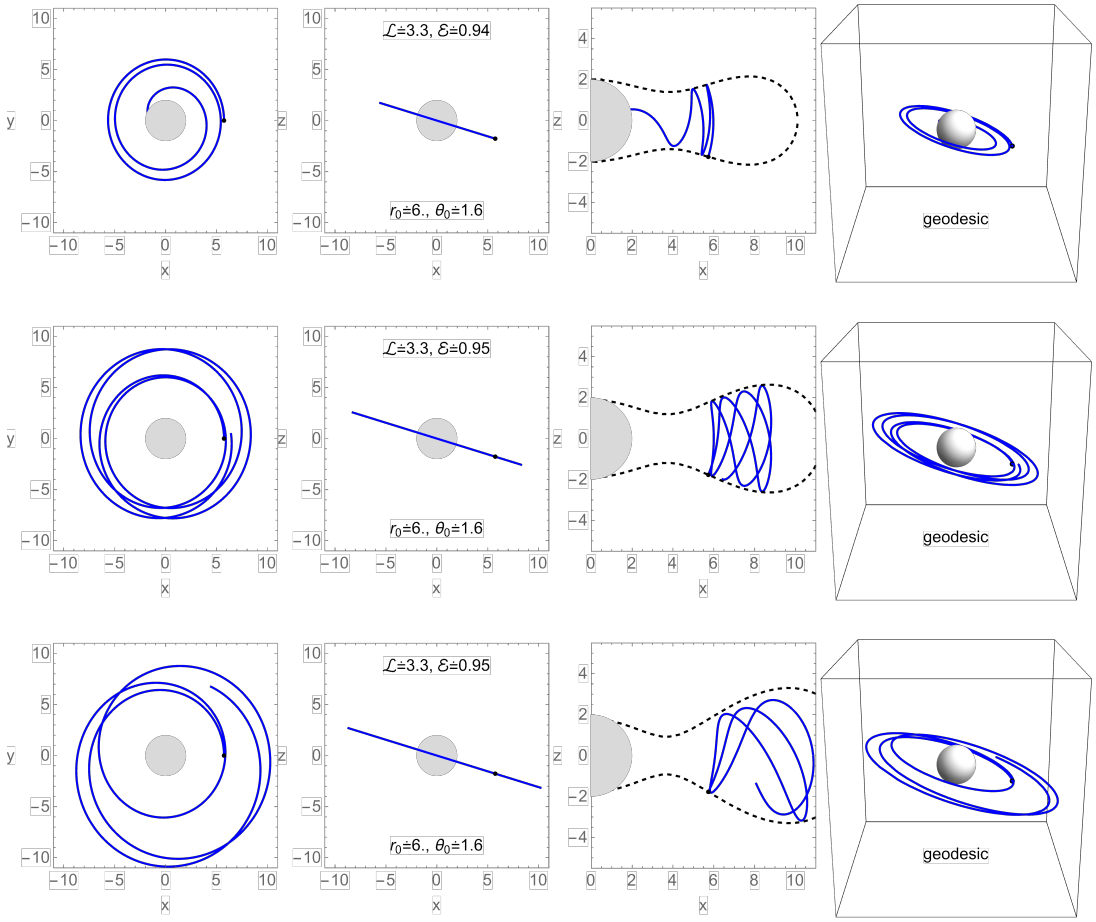


FIG. 2. The representation of the trajectory of non-equatorial motion for different values of a/M.

where $\kappa = 0$ for massless particles and $\kappa = 1$ for particles with mass. Thus the geodesic equations can be written in first–order form as [13–15]:

$$\dot{t} = \frac{E}{f(r)}, \qquad \dot{r} = \pm\sqrt{E^2 - f(r)\left(\kappa + \frac{L^2}{r^2}\right)}, \qquad \dot{\theta} = \frac{1}{r^2}\sqrt{L^2 - \frac{L_z^2}{\sin^2\theta}}, \qquad \dot{\varphi} = \frac{L_z}{r^2\sin^2\theta}. \tag{5}$$

These coupled differential equations are integrated numerically with given initial conditions (r_0, θ_0, ϕ_0) in order to obtain the particle trajectories[16]. Figures 1-3 show the dependencies of the trajectory of the test particle around our black hole in a vacuum.

Figure 1 shows the trajectories of a particle for different values of the parameter a/M at fixed values of angular momentum $\mathcal{L} = 3.3$ and energy $\mathcal{E} = 0.94$, with initial conditions $r_0 = 6$ in the equatorial case ($\theta = \pi/2$). The upper picture shows the Schwarzschild case with $a/M = 0$, in the middle panel $a/M = 0.1$ and in the bottom panel $a/M = 0.2$. It can be seen that, at low values of the parameter, the particle quickly loses stability and spirals down into the black hole (see upper panels). With intermediate values of the parameter, the trajectory becomes longer: the particle makes several revolutions around the object before falling. Finally, at higher values of the parameter (a/M is greater), the orbit stabilizes, and the particle can move in an almost circular orbit near the black hole, as can be seen in the lower graphs.

Figure 2 illustrates the non-equatorial case of motion with the corresponding parameter values ($a/M = 0$, $a/M = 0.1$, $a/M = 0.2$) presented from the top to the bottom images. In this case, an increase in the parameter value not only helps to stabilize the trajectory of the particles, but also leads to an increase in the radius of their rotation around the central object. This indicates that the a/M parameter has a significant effect on both the stability of the orbit and the geometry of motion, especially when deviating from the equatorial plane.

In Fig.3, we also consider the equatorial case with an increased angular momentum value $\mathcal{L} = 3.45$. Compared to the first figure, we can see that with a higher angular momentum value, the effect of the parameter increases. Thus, with an increase in the a/M parameter, not only the stabilization of motion is observed, but also the expansion of the particle's orbit. This highlights the

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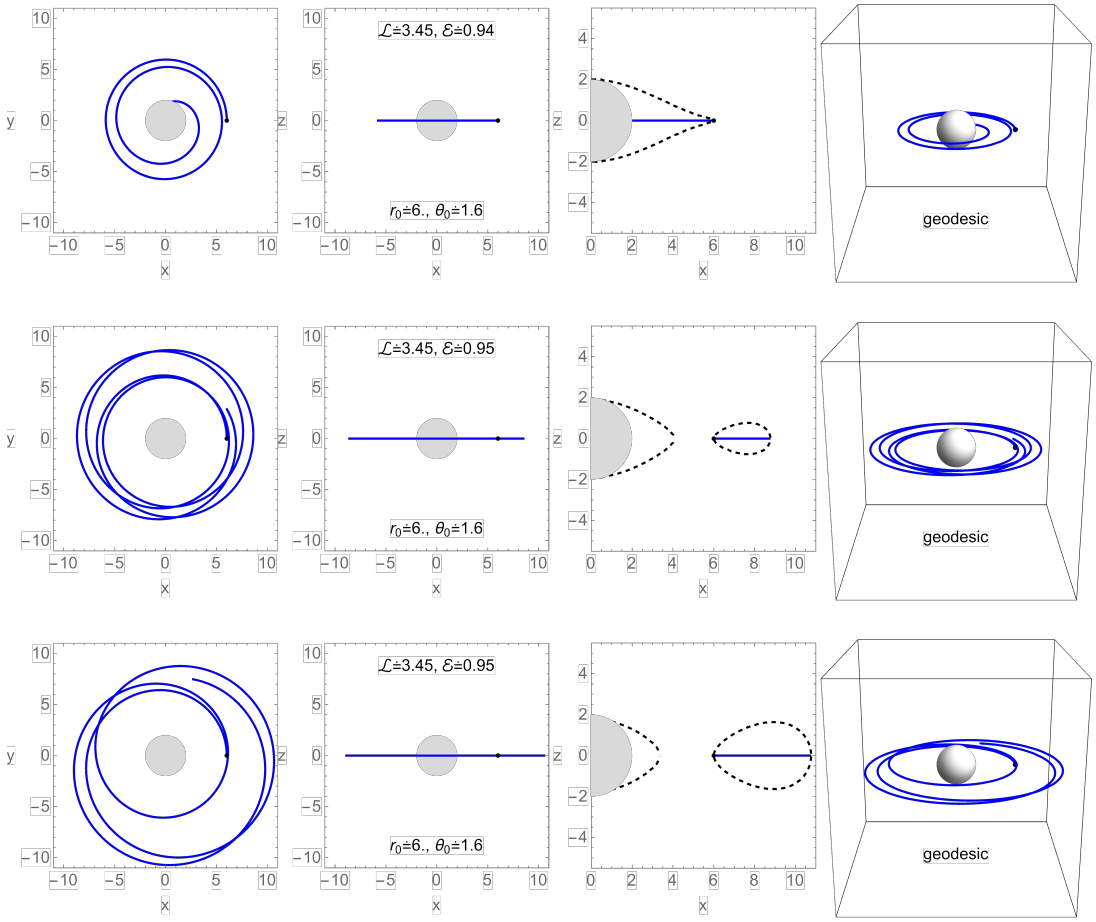


FIG. 3. Trajectory of particle in different values of parameter a/M for fixed $\mathcal{L} = 3.45$

important role of this parameter in shaping the spatial structure of the trajectory, especially in the case of non-equatorial motion.

III. CONCLUSION

In this study, we examined the properties of particle dynamics in motion around a Schwarzschild-like black hole. Specifically, we focused on observing the particle trajectory. We found that stability of motion orbit improves as the parameter of studied metric a/M increases. In the equatorial case, the influence of this parameter is non-visible on the radius of motion. But in the studied non-equatorial case, it can be seen that it contributes to an increase in the radius of the orbit of motion. The effect of angular momentum on the shape of the trajectory was also shown. According to our research, at high values of $\mathcal{L}$, the trajectory becomes more twisted and complex. In the case of Schwarzschild ($a/M = 0$), an increase in angular momentum led to an increase in rotations before falling into the central object.

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