Journal of Fundamental and Applied Research **Vol. 5**, Issue 3 (2025) **20250014**

Received: 30 July 2025 / Accepted: 26 September 2025
Published online: 29 September 2025
© 2025 The Author(s)

Davydov-Chaban model formalism with Davidson potential

Zebo Sobirova[1,*] and K. K. Shadmanov[2,†]

[1]*Chirchik State Pedagogical University, Chirchik, 111700, Uzbekistan*
[2]*Institute of Pharmaceutical Education and Research, Tashkent, 100114, Uzbekistan*

Abstract. The Davydov-Chaban model with the Davidson potential for describing the spectrum of excited collective states of heavy even-even nuclei is considered. At describing the energy levels of the ground-state-, γ- and β-bands of these states, the K-mixing effect created by model triaxial rotations is taken into account. The energy spectra of the ground-state-, γ- and β-bands of heavy even-even nuclei: ^{154}Sm ^{158,160,162}Gd, ^{160,166}Dy, ^{162,164,166}Er, ^{168,170}Hf, ^{168}Yb, ^{170}W, ^{228,230,232}Th, ^{232,234,236,238,240}U, ^{242,244}Pu are considered. The zigzag behavior of the ΔI=1 staggering effect in the spectrum of energy levels of γ-band of non-axial even-even nuclei: ^{166}Er, ^{230,232}Th, ^{234}U was determined.

Key words: quadrupole deformation, Bohr hamiltonian, collective states, effective triaxiality, Davidson potential, K-mixing effect, non-adiabaticity parameter, ΔI=1 staggering-effect.

I. INTRODUCTION

Article [1] presents the fundamentals of the collective model, and [2] firstly formulates the Bohr Hamiltonian or the collective Hamiltonian. This operator contains kinetic and potential terms in a set of variables describing the degree of deformation of the nuclear surface.

The problem of the solution of the Bohr Hamiltonian has been mainly tackled in two ways: direct solution of the second-order differential equation and use of algebraic techniques to exploit dynamical symmetries [3–11]. A novel interest has been raised by the possibility to give analytical solution at the critical point of shape phase transitions between various types of quadrupole deformed nuclear surfaces [3–6].

II. MODEL FORMALISM OR BOHR HAMILTONIAN

The energy levels of the ground-state-, γ- and β-bands of the heavy even-even nuclei are described by the Bohr hamiltonian containing five dynamic variables β ($\beta \geq 0$), γ ($0 \leq \gamma \leq \pi/3$), θ_1 ($0 \leq \theta_1 \leq 2\pi$), θ_2 ($0 \leq \theta_2 \leq \pi$), θ_3 ($0 \leq \theta_3 \leq 2\pi$) [1]:

$$\hat{H} = \hat{T}_\beta + \frac{1}{\beta^2}[\hat{T}_\gamma + \hat{T}_{rot}] + V(\beta, \gamma), \tag{1}$$

where

$$\hat{T}_\beta = -\frac{\hbar^2}{2B}\frac{1}{\beta^4}\frac{\partial}{\partial\beta}\left(\beta^4\frac{\partial}{\partial\beta}\right), \tag{2}$$

operator kinetic energy β-vibrations, here B–mass parameter.

$$\hat{T}_\gamma = -\frac{\hbar^2}{2B}\frac{1}{\sin(3\gamma)}\frac{\partial}{\partial\gamma}\left[\sin(3\gamma)\frac{\partial}{\partial\gamma}\right], \tag{3}$$

* Corresponding author: zeboabduqahhorova4@gmail.com
† shodman@mail.ru

operator kinetic energy γ-vibrations.

$$\hat{T}_{\rm rot} = \frac{1}{4B}\sum_{\lambda=1}^{3}\frac{\hat{I}_{\lambda}^{2}}{\sin^{2}\left(\gamma-\frac{2\pi}{3}\lambda\right)}, \tag{4}$$

rotational energy operator, here $\hat{I}_{\lambda}$ – components of the total angular momentum operator (λ = 1; 2; 3). $V(\beta,\gamma)$ – potential energy of β and γ-vibrations.

The general solution of the Schrödinger equation with Hamiltonian (1) is a very complicated [12]. Therefore are used different simplifications [3–13]. In Ref.[14] different solutions Schrödinger equation with Bohr hamiltonian (1) is considered for different modifications.

In present work we will develop Davydov-Chaban model formalism with Davidson potential. In this approximation the angular variable γ is replaced by its effective value, i. e. γ=$\gamma_{\rm eff}$.

III. DAVYDOV-CHABAN MODEL WITH DAVIDSON POTENTIAL

In Davydov-Chaban model we will write Bohr hamiltonian (1) in the following form [15]:

$$\hat{H} = \hat{T}_{\beta} + \frac{\hat{T}_{rot}}{\beta^{2}} + V(\beta), \tag{5}$$

where

$$\hat{T}_{\beta} = -\frac{\hbar^{2}}{2B}\frac{1}{\beta^{3}}\frac{\partial}{\partial\beta}\left(\beta^{3}\frac{\partial}{\partial\beta}\right), \tag{6}$$

$$\hat{T}_{\rm rot} = \frac{1}{4B}\sum_{\lambda=1}^{3}\frac{\hat{I}_{\lambda}^{2}}{\sin^{2}\left(\gamma_{\rm eff}-\frac{2\pi}{3}\lambda\right)}, \tag{7}$$

$V(\beta)$ – potential energy of β-vibrations.

We will write Schrödinger equation with Bohr hamiltonian (5):

$$\left[-\frac{\hbar^{2}}{2B}\frac{1}{\beta^{3}}\frac{\partial}{\partial\beta}\left(\beta^{3}\frac{\partial}{\partial\beta}\right)+\frac{1}{4B\beta^{2}}\sum_{\lambda=1}^{3}\frac{\hat{I}_{\lambda}^{2}}{\sin^{2}\left(\gamma_{\rm eff}-\frac{2\pi}{3}\lambda\right)}+V(\beta)-E_{In\tau}\right]\Psi_{In\tau}(\beta,\theta)=0. \tag{8}$$

The solution of the equation (8) we will find in the following form:

$$\Psi_{In\tau}(\beta,\theta) = F_{In\tau}(\beta)|IMK\rangle A_{IK}^{\tau},$$

where $|IMK\rangle$ – wave function rigid symmetry rotator in fhe form:

$$|IMK\rangle = \sqrt{\frac{2I+1}{16\pi^{2}(1+\delta_{K0})}}\left[D_{MK}^{I}+(-1)^{I}D_{M,-K}^{I}\right]$$

here $D_{MK}^{I}(\theta)$ – Wigner function; $\theta\equiv\theta_{1},\theta_{2},\theta_{3}$ – Euler angles describing the orientation of the nucleus in space; I – total angular momentum; M – projection of the total angular momentum onto the axial axis in the laboratory coordinate system; K – projection of the total angular momentum onto the axial axis in the internal coordinate system; δ_{0K} – Kronecker symbol; A_{IK}^{τ} – normalized wave functions of the rigid triaxial rotator [16].

Wave functions $|IMK\rangle A_{IK}^{\tau}$ satisfy equation for asymmetry rigid triaxial rotator:

$$\left[\frac{1}{2}\sum_{\lambda=1}^{3}\frac{\hat{I}_{\lambda}^{2}}{\sin^{2}\left(\gamma_{\rm eff}-\frac{2\pi}{3}\lambda\right)}-\varepsilon_{I\tau}\right]|IMK\rangle A_{IK}^{\tau}=0, \tag{9}$$

where $\varepsilon_{I\tau}$ – eigenvalues of the triaxial rotator [16]; the index τ enumerates the wave functions related to the same IK [16]. Triaxial quadrupole rotator model describes the structure of the lower levels of positive parity in the spectra of heavy even-even nuclei, which corresponds to a collective mechanism based on the K-mixing effect.

 Copyright © 2025

We will write equation (8) for $F_{In\tau}(\beta)$:

$$\left\{\frac{\partial^2}{\partial\beta^2}+\frac{3}{\beta}\frac{\partial}{\partial\beta}-\frac{\varepsilon_{I\tau}}{2\beta^2}-\frac{2B}{\hbar^2}[V(\beta)-E_{In\tau}]\right\}F_{In\tau}(\beta)=0, \tag{10}$$

In order to find the solution of Schrödinger equation (10) we choose $V(\beta)$ as Davidson potential [14, 17–20]:

$$V(\beta)=V_0\left(\frac{\beta}{\beta_0}-\frac{\beta_0}{\beta}\right)^2,$$

where β_0 – the position of the minimum of the potential, V_0 – potential energy of the ground state of an even-even nucleus.

IV. ENERGY LEVELS AND WAVE FUNCTIONS

Now we introduce following notations:

$$x=\frac{\sqrt{2BV_0}}{\hbar\beta_0}\beta^2; \qquad g=\frac{BV_0\beta_0^2}{\hbar^2}.$$

We obtain Schrödinger equation (10) in new variable:

$$\left[x\frac{\partial^2}{\partial x^2}+\frac{2\partial}{\partial x}-\frac{\varepsilon_{I\tau}+4g}{8x}-\frac{x}{4}+\tilde{E}\right]F_{In\tau}(x)=0, \tag{11}$$

where

$$\tilde{E}=\frac{2\sqrt{2g}}{4}+\frac{\sqrt{2g}}{4V_0}E_{In\tau}$$

Wave function $F_{In\tau}(x)$ we find in the following form:

$$F_{In\tau}(x)=e^{-\frac{x}{2}}x^s u_{In\tau}(x),$$

here

$$s=-0.5+\sqrt{\frac{2+\varepsilon_{I\tau}+4g}{8}}. \tag{12}$$

We obtain the confluent hypergeometric equation

$$xu''_{I\tau}(x)+(2s+2-x)u'_{In\tau}(x)+\left(-s-1+\tilde{E}\right)u_{In\tau}(x)=0. \tag{13}$$

Find energy spectrum:

$$E_{In\tau}=\frac{4V_0}{\sqrt{2g}}\left(n+s+1-\frac{\sqrt{2g}}{2}\right), \tag{14}$$

where n – quantum number β-vibrations. Introduce energy factor

$$\hbar\omega=\frac{4V_0}{\sqrt{2g}}.$$

We obtain final expression for energy spectrum:

$$E_{In\tau}=\frac{\hbar\omega}{2}\left(2n+1+\sqrt{\frac{2+\varepsilon_{I\tau}+4g}{2}}-\sqrt{2g}\right). \tag{15}$$

The sctructure of the bands of the excited collective states are determined by quantum number $(n;\tau)$. $(n;\tau)=(0;1)$ – ground state-band, $(n;\tau)=(0;2)$ – γ-band, $(n;\tau)=(1;1)$ – β-band.

Copyright © 2025

The resulting energy spectrum (15) differs from the similar Elliott-Evans-Park solution [17]. Basically an expression under the root.

The ground state energy with n=0, I=0, τ=1:

$$E_{001} = \hbar\omega \left(0.5 + 0.5\sqrt{1+2g} - \frac{\sqrt{2g}}{2}\right).$$

Energy levels of excited states:

$$\Delta E_{nI\tau} = E_{nI\tau} - E_{001} = \frac{\hbar\omega}{2}\left(2n + \sqrt{1 + \frac{\varepsilon_{I\tau}}{2} + 2g} - \sqrt{1+2g}\right). \tag{16}$$

Now we transform parameter g. If we taking into account, that

$$\mu^4 = \frac{1}{2g},$$

where μ – non-adiabacity parameter [16].

Energy levels of excited states in terms of μ:

$$\Delta E_{nI\tau} = \frac{\hbar\omega}{2}\left(2n + \sqrt{1 + \frac{\varepsilon_{I\tau}}{2} + \frac{1}{\mu^4}} - \sqrt{1 + \frac{1}{\mu^4}}\right). \tag{17}$$

The eigenfunctions of the equation (13) have the form:

$$F_{In\tau}(x) = Nx^s \exp\left(-\frac{x}{2}\right) F(-n, 2s+2, x). \tag{18}$$

where N is the normalization coefficient, $F(-n, 2s+2, x)$–confluent hypergeometric function.

Determine the normalization coefficients for the cases $n = 0$ and $n = 1$, respectively.

$$N_0 = \frac{\sqrt{2}}{\beta_0^2\mu^2}\frac{1}{\sqrt{\Gamma(2s+2)}}, \qquad N_1 = \frac{\sqrt{2}}{\beta_0^2\mu^2}\sqrt{\frac{2s+2}{\Gamma(2s+2)}}.$$

V. DISCUSSION

It is convenient to consider the ratio of excitation energies to the energy of the first excited level with spin 2^+ of the ground-state-band:

$$R_nI\tau = \frac{E_nI\tau}{E_021} \tag{19}$$

The results of calculating $R_{nI\tau}$ depending on the non-adiabaticity parameter g are presented in Fig. 1 for fixed values of the nonaxiality parameter γ_{eff} =10° and 20°.

The energies of the levels and the distances between the energy levels of the ground-state-, γ- and β-bands at a fixed γ_{eff} increase with increasing g. In addition, at $g \to 0$, the energy levels of the above-mentioned bands at a fixed γ_{eff} noticeably decrease and mix with each other, forming equidistant levels corresponding to spherical even-even nuclei.

At a fixed small values γ_{eff} ($\gamma_{\text{eff}} \leq 10°$), with increasing g, the equidistant spectrum is gradually violated and the energy levels of the γ- and β-bands noticeably increase compared to the ground-state-band, forming separate bands.

Moreover, the energy levels of the γ-band at small values of the triaxiality parameter γ_{eff} ($\gamma_{\text{eff}} \geq 10°$) lie higher than the energy levels of the β-band. In this case, the lower energy levels of even-even nuclei will be purely rotational and correspond to the spectra of deformed nuclei [21].

At large values of the triaxiality parameter ($\gamma_{\text{eff}} > 10°$), with increasing g, the energy levels of the γ-band fall below the energy levels of the β-band and mix with the energy levels of the ground-state-band and correspond to the spectra of transition even-even nuclei.

Consequently, depending on the values of the parameters g and γ_{eff}, the shape of the deformed even-even non-axial atomic nucleus changes from spherical to ellipsoidal. Thus, it becomes possible to describe in a unified way the spectrum of excited state levels of spherical, transition, and non-spherical even-even nuclei with two parameters g and γ_{eff}, which are determined from a comparison of the theoretical values of the energies of the ground-state- and γ-bands with the experimental ones [22].

 Copyright © 2025

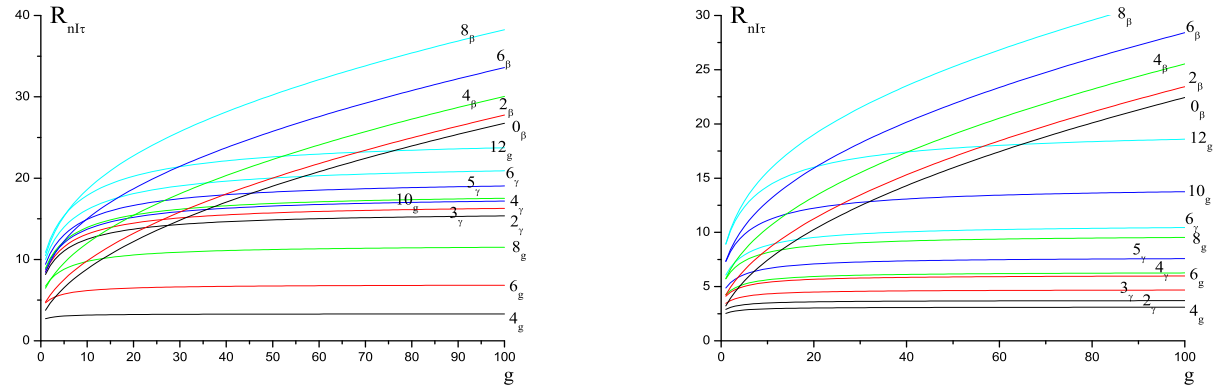


FIG. 1. Dependence of the ratio of the energy levels of excited collective states of the ground-state-, γ- and β-bands to the energy of the first excited level with spin 2^+ of the ground-state-band from parameter g at the fixed values of the triaxiality parameter $\gamma_{\rm eff}$ = 10° and 20°.

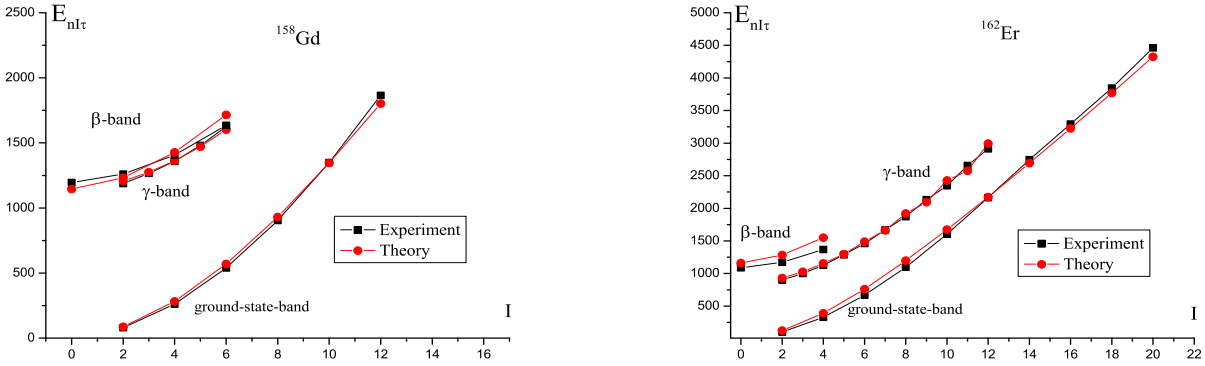


FIG. 2. The behavior of the theoretical and experimental [22] energy levels of the ground-state-, γ- and β-bands for nuclei: ^{158}Gd (left) with theoratical parameters: $\hbar\omega$=572.8 keV, g=23.5, $\gamma_{\rm eff}$=10°, RMS=34 keV and ^{162}Er (right) with theoratical parameters: $\hbar\omega$=580.5 keV, g=12.7, $\gamma_{\rm eff}$=13.5°, RMS=75.8 keV.

VI. RESULTS

In the present work we consider the energy levels of the ground-state-, γ- and β-bands of the non-axial even-even nuclei. These energy levels are described by adjusting three model parameters: energy factor $\hbar\omega$ (in keV), non-adiabaticity parameter g (dimensionless), triaxiality parameter $\gamma_{\rm eff}$ (in degrees).

Within the framework of the proposed model, the energy spectrum of excited collective states of the following heavy even-even nuclei is considered: ^{154}Sm ^{158,160,162}Gd, ^{160,166}Dy, ^{162,164,166}Er, ^{168,170}Hf, ^{168}Yb, ^{170}W, ^{228,230,232}Th, ^{232,234,236,238,240}U and ^{242,244}Pu. Below are some of these results.

The behavior of the theoretical and experimental [22] energy levels of the ground-state-, γ- and β-bands for nuclei: ^{158}Gd, ^{162,166}Er, ^{166}Dy, ^{168}Hf, ^{168}Yb, ^{170}W, ^{230,232}Th, ^{234}U are presented in Figs. 2-6. The values of the fitting parameters and the standard deviation of the energy levels are given in the captions.

VII. ΔI=1 STAGGERING-EFFECT

Taking into account the effective triaxiality leads to the appearance of new bands. These bands are considered as rotational,

Copyright © 2025

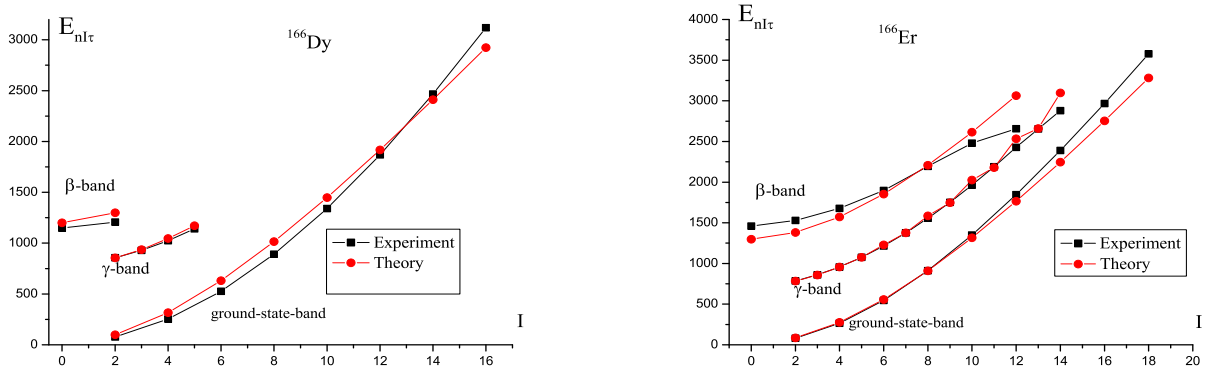


FIG. 3. The same as in Fig. 2, but for nuclei: ^{166}Dy (left) with $\hbar\omega$=600 keV, g=21.7, $\gamma_{\rm eff}$=12.8°, RMS=83.4 keV and ^{166}Er (right) with $\hbar\omega$=648.7 keV, g=34.4, $\gamma_{\rm eff}$=12.6°, RMS=127.1 keV.

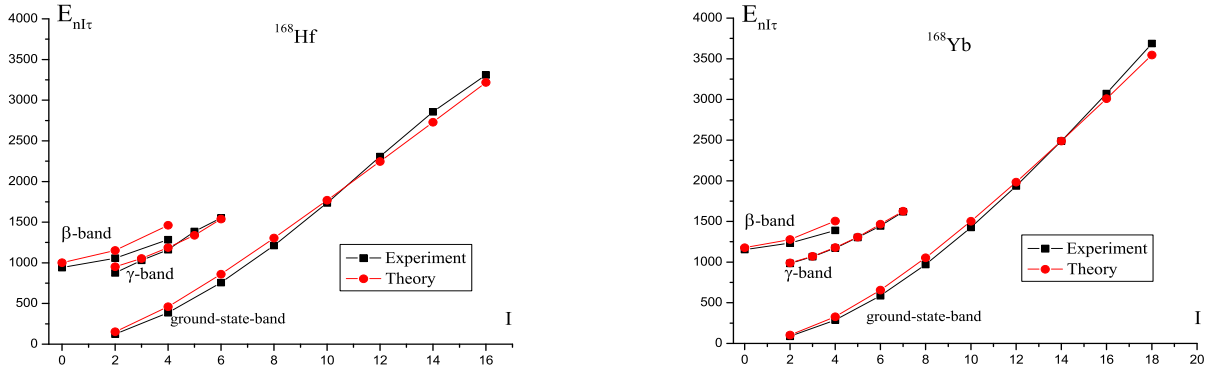


FIG. 4. The same as in Fig. 2, but for nuclei: ^{168}Hf (left) with $\hbar\omega$=500 keV, g=5.9, $\gamma_{\rm eff}$=14.3°, RMS=82.3 keV and ^{168}Yb (right) with $\hbar\omega$=588 keV, g=18.6, $\gamma_{\rm eff}$=12°, RMS=58.7 keV.

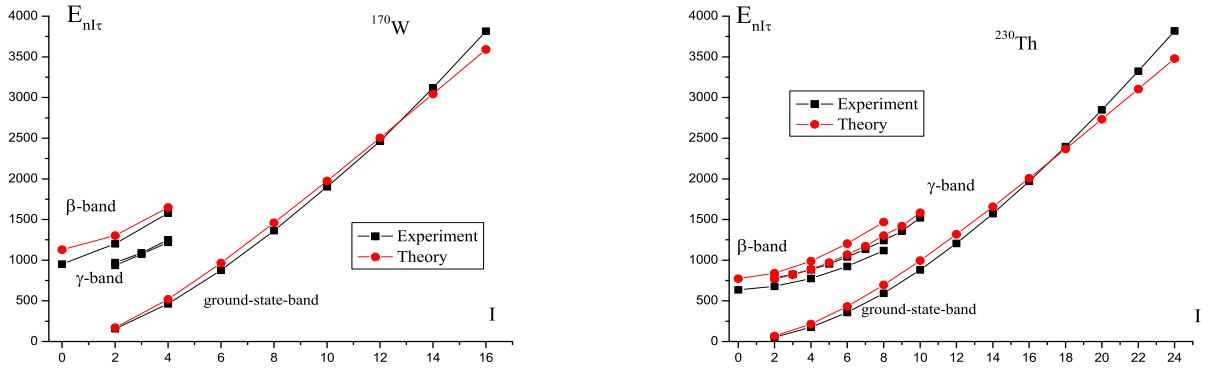


FIG. 5. The same as in Fig. 2, but for nuclei: ^{170}W (left) with with $\hbar\omega$=565 keV, g=6.1, $\gamma_{\rm eff}$=15.1°, RMS=97.4 keV and ^{230}Th (right) with $\hbar\omega$=386.7 keV, g=18.2, $\gamma_{\rm eff}$=11°, RMS=114 keV.

Copyright © 2025

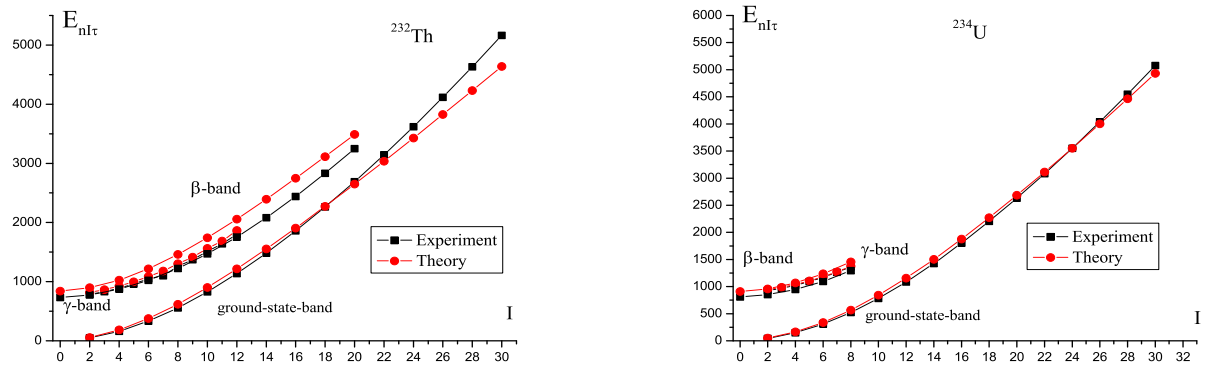


FIG. 6. The same as in Fig. 2, but for nuclei: ^{232}Th (left) with $\hbar\omega$=420.5 keV, g=30, $\gamma_{\rm eff}$=10°, RMS=184.4 keV and ^{234}U (right) with $\hbar\omega$=454.2 keV, g=47.1, $\gamma_{\rm eff}$=8.6°, RMS=84.9 keV.

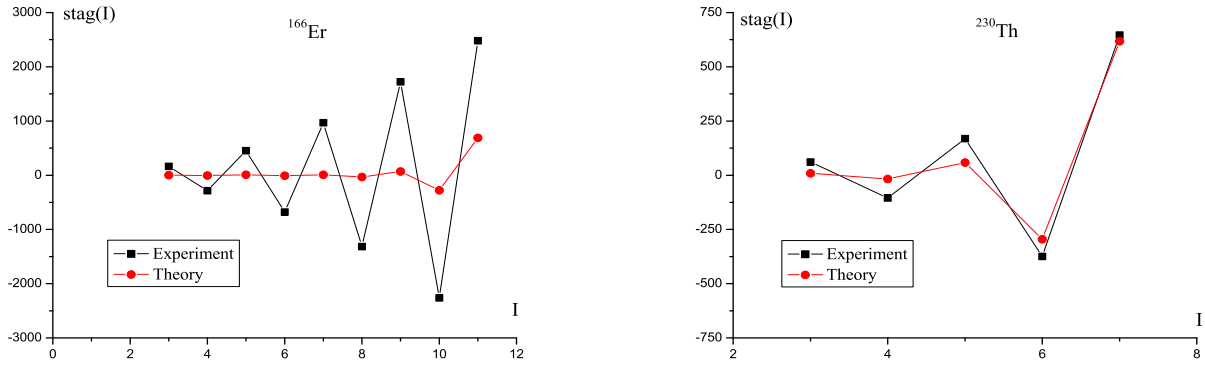


FIG. 7. The behavior of the theoretical and experimental [22] ΔI=1 staggering effect in the spectrum of energy levels of γ-band for nuclei: ^{166}Er (left) and ^{230}Th (right).

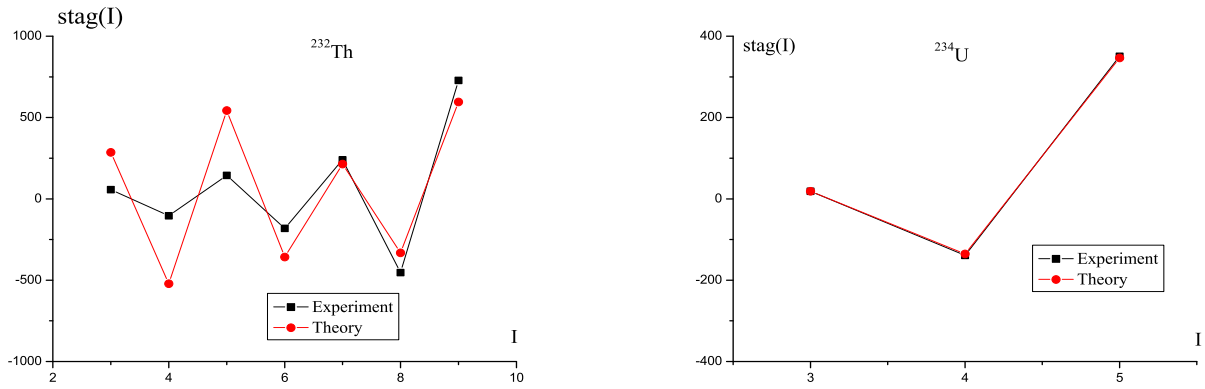


FIG. 8. The same as in Fig. 7, but for nuclei: ^{232}Th (left) and ^{234}U (right).

Copyright © 2025

but in reality they are complex excitations of the rotational-γ-vibrational type [15]. More detailed information about the structure of γ-band can be obtained on the basis of the so-called ΔI=1 staggering analysis which refers to the mutual displacement of the odd- and even- values of the angular momentum levels in the γ-band. The effect can be quantified through the fifth discrete derivative of the energy as a function of the angular momentum [23]:

$$Stag(I) = 6\Delta E(I) - 4\Delta E(I-1) - 4\Delta E(I+1) + \Delta E(I+2) + \Delta E(I-2), \tag{20}$$

with $\Delta E(I) = E(I+1) - E(I)$. This quantity vanishes if the even- and the odd- angular momentum levels form a single rotation band and exhibit different kinds of zig-zagging (staggering) patterns when the two sequences are shifted from the one another. The behavior of the theoretical and experimental ΔI=1 staggering effect in γ-band for nuclei: ^{166}Er, ^{230,232}Th, ^{234}U are presented in Figs. 7-8.

VIII. CONCLUSION

In the proposed work, the Davydov-Chaban model with the Davidson potential for surface deformations is developed. An analytical solution of the Schrödinger equation with a Hamiltonian describing deformed non-axial even-even nuclei with quadrupole deformation is obtained for the Davidson potential energy of surface oscillations. Explicit expressions for the energy spectrum and wave functions of excited collective states of non-axial even-even nuclei are found. The energy spectra of the ground-state-, γ- and β-bands of heavy even-even nuclei are considered, by taken into accouting the K-mixing effect created by the model triaxial rotations for above-mentioned states. The energy spectra of the ground-state-, γ- and β-bands of heavy even-even nuclei: ^{158}Gd, ^{162,166}Er, ^{166}Dy, ^{168}Hf, ^{168}Yb, ^{170}W, ^{230,232}Th, ^{234}U are presented in this paper.

A comparison with experimental data was performed, which showed good agreement between theoretical and experimental data. The relevance of the proposed non-adiabatic collective model with quadrupole deformation taking into account the effective non-axiality of the shape of heavy even-even nuclei was confirmed.

The zigzag behavior of the ΔI=1 staggering effect in the spectrum of energy levels of γ-band of non-axial even-even nuclei: ^{166}Er, ^{230,232}Th, ^{234}U was determined. The obtained behavior of the ΔI=1 staggering effect reproduces well the fine structure of the positive parity spectra in the listed nuclei.

REFERENCES

[1] A. Bohr Dan. Mat. Fys. Medd. 26, No. 14 (1952).
[2] A. Bohr, B. R. Mottelson Mat. Fys. Medd. Dan. Vid. Selsk. 27, No. 16 (1953).
[3] A. V. Turbiner *Physics Reports.* **642** (2016) 1.
[4] P. Buganu and R. Budaca *Physical Review C.* **91** (2015) 014306.
[5] P. Buganu and R. Budaca *Journal of Physics G: Nuclear and Particle Physics.* **42** (2015) 105106.
[6] R. Budaca, A. I. Budaca and P. Buganu *Journal of Physics G: Nuclear and Particle Physics.* **46** (2019) 125102.
[7] M. Capak, D. Petrellis, B. Gonul and D. Bonatsos *Journal of Physics G: Nuclear and Particle Physics.* **42** (2015) 95102.
[8] H. Sobhani, A. N. Ikot, Hassanabadi H. *The European Physical Journal.* **132** (2017) 240.
[9] H. Sobhani et al., *The European Physical Journal A.* **56** (2020) 29.
[10] M. M. Hammad, A. Sh. Yaqut, M. A. Abdel-Khalek, S. B. Doma *Nuclear Physics A.* **1015** (2021) 122307.
[11] R. Budaca and P. Buganu *Physical Review C.* **100** (2019) 044613.
[12] Dennis Bonatsos, D. Lenis, D. Petrellis *Romanian Reports in Physics.* **59** (2007) 273.
[13] M. S. Nadirbekov and G. A. Yuldasheva *IJMP E.* **23** (2014) 1450034.
[14] L. Fortunato and A. Vitturi *Journal of Physics G: Nuclear and Particle Physics.* **29** (2003) 1341.
[15] A. S. Davydov and A. A. Chaban *Nuclear Physics.* **20** (1960) 499.
[16] A. S. Davydov *Excited states of atomic nuclei.* [in Russian]. (Atomizdat, Moscow, 1967).
[17] J. P. Elliott, J. A. Evans and P. Park *Physics Letters B.* **169** (1986) 309.
[18] Dennis Bonatsos, E. A. McCutchan, N. Minkov, R. F. Casten, P. Yotov, D. Lenis, D. Petrellis and I. Yigitoglu *Physical Review C.* **76** (2007).
[19] I. Yigitoglu and D. Bonatsos *Physical Review C.* **83** (2011).
[20] D. J. Rowe and C. J. Bahri *Phys. A: Math. Gen.* **31** (1998) 4947.
[21] M. S. Nadirbekov, O. A. Bozarov *Physics of Atomic Nuclei.* **99** (2016) 461.
[22] http: // www.nndc.bnl.gov/ensdf.
[23] N. Minkov, S. Drenska, P. Raychev, R. Roussev and D. Bonatsos *Phys. Rev. C.* **63** (2001) 044305.

 Copyright © 2025