Journal of Fundamental and Applied Research **Vol. 4**, Issue 2 (2024) **20240007**

Received: 30 May 2024 / Accepted: 28 June 2024
Published online: 28 June 2024
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Deriving the equations of particle motion under an external electromagnetic field in a cylindrical coordinate system near a compact gravitational object

Sardor Tojiev,[1,2,*] Abdullo Hakimov,[3,†] Nodirbek Bahromov,[4,‡] Munisa Boltatoshova,[4,§] and Azamkhan Rakhmatov[4]

[1] *Ulugh Beg Astronomical Institute, Astronomy St. 33, Tashkent 100052, Uzbekistan*
[2] *Institute of Fundamental and Applied Research, National Research University TIIAME, Kori Niyoziy 39, Tashkent 100000, Uzbekistan*
[3] *Department of Applied Mathematics, Samarkand International University of Technology, 270 Spitamen Avenue, Samarkand 140100, Uzbekistan*
[4] *National University of Uzbekistan, Tashkent 100174, Uzbekistan*

Abstract. The motion of particles in the presence of gravitational and electromagnetic fields is a fundamental topic in theoretical physics, with applications ranging from astrophysics to plasma physics. This paper reviews the derivation of particle motion equations in a cylindrical coordinate system when influenced by external electromagnetic fields, particularly near compact gravitational objects such as black holes and neutron stars. Using the principles of classical mechanics and electromagnetic theory, we explore the Lorentz force, the influence of curvature in spacetime, and the appropriate mathematical tools for analysis. The implications of these equations in understanding the dynamics of charged particles in extreme environments are also discussed.

Key words: particle motion, electromagnetic fields, gravitational influence, cylindrical coordinate system, compact gravitational objects, Lorentz force, Schwarzschild metric.

I. INTRODUCTION

The study of particle motion in gravitational and electromagnetic fields has long been a cornerstone of modern physics. When considering compact gravitational objects, such as black holes or neutron stars, this interplay becomes particularly intricate due to the strong gravitational fields and the resulting curvature of spacetime. Meanwhile, external electromagnetic fields can significantly alter particle trajectories, especially if those particles are charged.

In cylindrical coordinates, the symmetry of many systems simplifies the mathematical treatment of motion. This coordinate system is particularly useful in scenarios where the geometry is inherently cylindrical, such as in the vicinity of rotating black holes or in plasma confinement systems. The objective of this paper is to derive the equations of motion for charged particles under the influence of both gravitational and electromagnetic fields, specifically in a cylindrical coordinate framework.

The Lorentz force law, which governs the motion of charged particles in electromagnetic fields, plays a crucial role in these derivations. In conjunction with Einstein's general relativity, this leads to a comprehensive understanding of particle dynamics near compact gravitational bodies. This review will cover the key principles, methodologies, and results associated with deriving these equations, while highlighting the significance of the findings for both theoretical and practical applications.

II. METHODOLOGIES

In the study of particle motion within electromagnetic fields and under gravitational influence, a comprehensive understanding of the governing principles is paramount. This interplay is especially crucial when investigating the behavior of charged particles near compact gravitational objects, such as black holes and stars. To analyze these complex interactions, we employ a variety of

* Corresponding author: sardor@astrin.uz
† abdullo@astrin.uz
‡ nodirbahromov9@gmail.com
§ boltatoshovamunisa@gmail.com

methodologies that integrate theoretical frameworks and computational techniques. By leveraging advanced mathematical models, numerical simulations, and analytical approaches, we aim to gain deeper into the dynamics of particle motion, particularly in extreme astrophysical environments. The following methodologies are:

1. Mathematical Modeling: We utilize mathematical models to describe the motion of particles in both electromagnetic and gravitational fields. The Lorentz force equation is employed to quantify the influence of electromagnetic fields on charged particles, while the Schwarzschild metric serves as the foundational description of spacetime around non-rotating black holes. This dual approach allows for a comprehensive understanding of particle dynamics in varying contexts, enabling the formulation of precise predictions regarding particle trajectories.

2. Cylindrical Coordinate System: To simplify the analysis of particle motion in specificies, we adopt a cylindrical coordinate system. This is particularly beneficial when examining trajectories in systems with cylindrical symmetry, such as charged particle beams in accelerators or plasma confinement devices. The conversion of coordinates facilitates easier integration and manipulation of the governing equations, thereby enhancing our ability to visualize particle interactions and the effects of external fields.

3. Numerical Simulations: Given the complexity of particle interactions in strong gravitational and electromagnetic fields, numerical simulations are employed to visualize and predict particle trajectories. Techniques such as the Runge-Kutta method are often used to solve ordinary differential equations, allowing for the exploration of various initial conditions and parameters that characterize the motion of particles. These simulations provide invaluable insights into the of orbits and the potential for chaotic behavior in dynamic systems.

4. Perturbation: In scenarios where exact solutions are intractable, perturb theory offers a means to approximate the behavior of particles in weak gravitational fields or non-linear electromagnetic fields. This method is effective in analyzing small deviations from known solutions, providing insights into the stability and dynamics of particle systems over time. By applying perturbative, we can explore how minor changes in field strength or particle charge can significantly impact particle trajectories.

5. Spectral Analysis: By employing spectral analysis techniques, we can investigate the frequencies and energy levels associated with particle motion in fields. This approach is particularly useful in determining the resonant conditions under which particles may experience enhanced interactions, leading to potential astrophysical phenomena such as acceleration or emission of radiation. Spectral methods allow us to quantify how electromagnetic and gravitational forces interact to influence particle energy states and transition rates.

6. Analytical Solutions: When possible, we derive analytical solutions to simpler cases of particle motion, allowing for a clearer understanding of underlying physical principles. These solutions serve as benchmarks for numerical methods and help identify regions of parameter space where more complex behaviors may emerge.

By integrating these methodologies, we aim to create a holistic framework for particle motion in the challenging environments of compact gravitational objects and strong electromagnetic fields, paving the way for novel astrophysical insights and applications.

III. DERIVING THE EQUATIONS OF PARTICLE MOTION UNDER AN EXTERNAL ELECTROMAGNETIC FIELD IN A CYLINDRICAL COORDINATE SYSTEM NEAR A COMPACT GRAVITATIONAL OBJECT

In astrophysical environments characterized by the presence of compact gravitational objects such as black holes, the particle motion is profoundly affected not only by the strong gravitational field but also by external electromagnetic fields. This section focuses on deriving the equations that govern the motion of such particles near a compact gravitational object, specifically in the context of a cylindrical coordinate system - the Boyer-Lindquist coordinates, which are well suited to the Kerr metric describing rotating black holes [1, 2].

The motion of particles in these extreme environments is inherently complex, as it is determined by the interaction of gravitational and electromagnetic forces in a curved spacetime. Standard Newtonian dynamics are insufficient in this context, and instead a relativity treatment is required. This involves using the geodesic equations of general relativity, which describe how particles move along trajectories defined by the curvature of spacetime. However, when external electromagnetic fields are present, these equations must be modified to account for the additional forces acting on the particles.

In the Boyer-Lindquist coordinate system, which is consistent with the rotational symmetry of the Kerr black hole, these modifications present unique challenges and opportunities. The equations of motion must be adapted to account for the cylindrical nature of the coordinates, making them more complex, but also more representative of the actual physical conditions near a rotating black hole.

The goal of this section is to derive these modified equations of motion in a methodical manner. We begin by revisiting the geodesic equations in the context of curved spacetime and then introduce modifications due to external electromagnetic

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fields. The goal is to develop a set of equations that accurately describe the motion of charged particles in the gravitational and electromagnetic fields characteristic of the region surrounding a compact gravitational object. These equations are important for understanding a number of astrophysical phenomena, including the dynamics of accretion disks, jet formation, and particle acceleration processes near black holes.

Next, we demonstrate the study of the motion of charged particles around a black hole under conformal gravity in the presence of an external uniform magnetic field.

The Hamilton-Jacobi equation for a test particle with mass m and charge q can be written as follows:

$$g^{\mu\nu}\Big(\frac{\partial \mathcal{S}}{\partial x^{\mu}} - qA_{\mu}\Big)\Big(\frac{\partial \mathcal{S}}{\partial x^{\nu}} - qA_{\nu}\Big) = -m^2 \,. \tag{1}$$

The solution of equation (1) can be sought in the following form:

$$\mathcal{S} = -Et + l\phi + \mathcal{S}_r(r) + \mathcal{S}_{\theta}(\theta) \,, \tag{2}$$

where $\mathcal{E} = E/m$ and $\mathcal{L} = l/m$ are the specific energy and specific angular momentum of the test particle, respectively.

It is convenient to consider the particle motion on the equatorial plane, where $\dot{\theta} = 0$ ($p_{\theta} = 0$). Then we can write:

$$\dot{r}^2 + V_{\text{eff}}(r) = \mathcal{E}^2 \,, \tag{3}$$

where the effective potential is:

$$V_{\text{eff}}(r) = f(r)\Big[S(r) + \Big(\frac{\mathcal{L}}{r} - \omega_{\text{B}} r S(r)\Big)^2\Big] \tag{4}$$

and $\omega_{\text{B}} = eB/2m$ is the magnetic coupling parameter, or the so-called cyclotron frequency, which characterizes the interaction between a charged particle and a magnetic field. The effective potential is invariant under the following transformations: $(\mathcal{L}, -\omega_{\text{B}}) \longleftrightarrow (-\mathcal{L}, \omega_{\text{B}})$, where the Lorentz force acting on a charged particle is repulsive and has the same direction as the centrifugal force, and $(\mathcal{L}, \omega_{\text{B}}) \longleftrightarrow (-\mathcal{L}, -\omega_{\text{B}})$, where the Lorentz force is attractive and has the same direction as the gravitational force. Below we analyze the effective potential (4) for positive angular momenta of the particle and positive or negative magnetic coupling parameters.

IV. ENCODING OF THE MODULE FOR CALCULATING THE FIELD VALUE AT GRID NODES BASED ON THE PARTICLE DISTRIBUTION.

The study of electromagnetic fields in curved spacetime is a cornerstone of our understanding of various astrophysical phenomena, especially those that occur in extreme environments near black holes. In such regions, traditional Newtonian notions of space and time are significantly modified by strong gravitational fields, requiring a general relative concept to accurately describe electromagnetic interactions [3, 4].

Near a black hole, space and time are not only curved but also interconnected, profoundly influencing the behavior of electromagnetic fields. Calculating the electric (E) and magnetic (B) fields in this curved spacetime is important for several reasons:

- *Understanding accretion disks:* The dynamics of accretion disks around black holes depend heavily on electromagnetic forces. Accurately modeling the E and B fields helps predict the behavior of the disk, including the inflow rates and energy emission.
- *Predicting jet formation:* Many black holes are known to eject massive jets of plasma at speeds close to the speed of light. The formation and maintenance of these jets is closely related to the configuration of the magnetic fields, which can only be understood through accurate calculations in the framework of curved spacetime.
- *Testing general relativity:* These calculations also provide a field for testing the predictions of General Relativity, especially in regions with strong gravitational influences.
- *Astrophysical Phenomena:* From gravitational lensing to radiation propagation, the behavior of electromagnetic fields near a black hole influences a number of observed phenomena, making their study important for astrophysical research.

A specific focus of this study is the environment around a black hole, especially the regions close to and inside the accretion disk. Black holes, with their extreme gravitational fields, provide a unique laboratory for studying physics under conditions unattainable on Earth. Here, the dynamics of many particles around and inside the accretion disk are determined by both gravitational and electromagnetic interactions. Modeling these dynamic processes requires not only a deep understanding of general relativity, but also a sophisticated approach to electromagnetism in curved spacetime [5]. This section aims to clarify these complex interactions by providing a methodology for calculating the E and B fields at a given point in space, characterized by the Boyer-Lindquist coordinates (r, θ) in the vicinity of a black hole. The main challenge is to adapt Maxwell's equations to the conditions of curved spacetime and to apply them to model the behavior of particles in the accretion disk, which is fundamental to understanding the broader astrophysical processes that take place.

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Maxwell's Equations in Curved Spacetime

Maxwell's equations are the basis of classical electrodynamics, describing how electric and magnetic fields propagate and interact with matter. In the context of general relativity, where spacetime is curved by massive objects, these equations must be reformulated to take into account the effects of curvature. This subsection provides an introduction to Maxwell's equations and their generalizations in general relativity [6, 7].

In flat spacetime, Maxwell's equations are traditionally expressed in the following differential form [8]:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{7}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{8}$$

Here $\mathbf{E}$ and $\mathbf{B}$ represent are the electric and magnetic fields, respectively, ρ is the charge density, $\mathbf{J}$ is the current density, ϵ_0 is the vacuum permittivity, and μ_0 is the vacuum magnetic permeability.

In the curved spacetime described by general relativity, Maxwell's equations must be expressed in a covariant form that remains valid in any coordinate system. This is achieved by replacing partial derivatives with covariant derivatives and using tensor notation [9]. The equations become:

$$\nabla_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \tag{9}$$

$$\nabla_{[\alpha} F_{\beta\gamma]} = 0 \tag{10}$$

Here $F^{\mu\nu}$ is the electromagnetic field tensor, second-rank and antisymmetric, including both $\mathbf{E}$ and $\mathbf{B}$ fields. J^ν is the four-dimensional current density, and ∇_μ denotes the covariant derivative. The square brackets around the indices denote the antisymmetric part of the tensor, ensuring consistency with the sourceless nature of the magnetic field.

The curvature of spacetime changes the paths along which electromagnetic waves propagate. This effect, absent in flat spacetime, is described by the Christoffel symbols that appear in the covariant derivative calculations. These modifications reflect the influence of gravity on electromagnetic phenomena, a key concept in the study of black holes and other astrophysical objects where gravitational fields are extremely strong [10, 11].

In the following subsections, we will consider how these equations apply in the context of a black hole described by the Kerr metric, and how they are used to calculate the electric and magnetic fields in the vicinity of the accretion disk.

Kerr Metric and Boyer-Lindquist Coordinates The Kerr metric is a solution of Einstein's equations of general relativity that describes the geometry of spacetime around a rotating black hole. Unlike the simpler Schwarzschild metric, which describes non-rotating black holes, the Kerr metric takes into account angular momentum effects.

The Kerr metric is characterized by the mass M of the black hole and its angular momentum J or, equivalently, the spin parameter $a = J/M$. In Boyer-Lindquist coordinates the metric is expressed as follows [12, 13]:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \tag{11}$$

$$+ \left(r^2 + a^2 + \frac{2Ma^2 r \sin^2\theta}{\Sigma}\right) \sin^2\theta d\phi^2 \tag{12}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. This form of metric, although complicated, provides significant advantages in analyzing spacetime around rotating black holes.

The Boyer-Lindquist coordinates (t, r, θ, ϕ) are a generalization of spherical coordinates to the Kerr metric. They are particularly useful in studying black holes for several reasons:

- *Separability:* In these coordinates, the equations describing the motion of particles and fields are more easily separated, facilitating analytical and numerical solutions.
- *Physical understanding:* While r and θ have similar values as in spherical coordinates, the t coordinate represents time as measured by a distant observer and ϕ represents the azimuthal angle around the black hole's axis of rotation. This similarity to spherical coordinates provides some physical insight.

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- *Applications in astrophysics:* Boyer-Lindquist coordinates are particularly useful for describing phenomena such as accretion disks and jet formation around black holes, where the effects of rotation and strong gravity are significant.

The choice of Boyer-Lindquist coordinates is not only a mathematical convenience, but also a practical tool in the astrophysical study of rotating black holes and their environments.

Electrodynamics near black holes The study of electrodynamics near black holes is a field that combines the principles of electromagnetism with the complex geometry of spacetime described by general relativity. The vicinity of a black hole presents unique challenges and aspects that are significantly different from electrodynamics in flat space [14, 15].

In the strong gravitational fields near black holes, several unique aspects of electrodynamics become apparent:

- *Curved spacetime effects:* The curvature of spacetime around a black hole affects the propagation of electromagnetic fields. Unlike flat space, the trajectories of electromagnetic waves are bent by the curvature, leading to phenomena such as gravitational lensing.
- *Coordinate dragging:* Rotating black holes described by the Kerr metric exhibit a coordinate dragging effect, where spacetime itself is twisted. This effect has profound consequences for the behavior of electromagnetic fields, changing their orientation and propagation.
- *Time dilation:* Time dilation becomes significant near a black hole. This affects the observation and measurement of electromagnetic phenomena, since time flows differently for observers at different distances from the black hole.

Studying the electrodynamics near black holes involves several challenges:

- *Complex field equations:* Maxwell's equations in curved spacetime include additional terms and complications due to the curvature of spacetime. Solving these equations requires advanced mathematical methods and numerical techniques.
- *Extreme conditions:* The extreme gravitational and magnetic fields near black holes are beyond the scope of current physical theories and computational models. Accurate modeling of these conditions is essential for understanding phenomena such as the dynamics of accretion disks and jet formation.
- *Interaction with gravitational fields:* The interaction between the electromagnetic and gravitational fields near a black hole is complex and nonlinear. This interaction is key to understanding various high-energy astrophysical processes.

Studying the electrodynamics near black holes not only deepens our understanding of these mysterious objects, but also provides unique scientific insights into the fundamental laws of physics under extreme conditions. Therefore, it remains an active area of research in astrophysics and theoretical physics.

Computational Approach

The computational approach to solving Maxwell's equations near a black hole, especially in the context of curved spacetime, involves several steps and considerations. This approach is designed to balance the need for physical accuracy with the limitations of computational resources [16].

Solving Maxwell's equations in curved spacetime requires adapting conventional electromagnetic theory to the geometric complications introduced by general relativity. The basic approach involves solving these equations numerically, taking into account the curvature of spacetime near the black hole [17, 18].

The electromagnetic field tensor $F^{\mu\nu}$ is the central element of this approach, describing the electric and magnetic fields in a covariant formulation. The equations are solved using a discretized version of spacetime, typically using finite difference or finite element methods to approximate the derivatives. This numerical approach allows the electric and magnetic fields to be calculated at discrete points near the black hole.

Simplifications and Assumptions Several simplifications and assumptions are made in developing a computational model to solve these equations:

Axisymmetric Space: The model assumes an axisymmetric space, typically represented by the Kerr metric. This assumption simplifies the complexity of the spacetime geometry, making the computational problem more tractable.

Boyer-Lindquist Coordinates: The use of Boyer-Lindquist coordinates provides a suitable framework for discretizing the spacetime around a rotating black hole. These coordinates facilitate the separation of variables and simplify the computational implementation of the field equations.

Linearized Field Equations: In some cases, the field equations are linearized to reduce the computational complexity. Although this approach may not capture all nonlinear interactions, it provides a reasonable approximation for many scenarios, especially when the field intensities are not excessively large.

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Boundary conditions: Appropriate boundary conditions are applied at the event horizon and at large distances from the black hole. These conditions are chosen to reflect physical expectations such as the absence of radiation from the event horizon and the asymptotic behavior of the fields at large distances [19].

The combination of these computational strategies and simplifications provides a framework for studying the behavior of electromagnetic fields in the extreme environment near black holes. Although some subtleties of the exact solutions may be missed, this approach offers a practical balance between computational feasibility and physical realism.

Calculating the metric tensor and Christoffel symbols Calculating the metric tensor and Christoffel symbols is a critical step in solving Maxwell's equations in curved spacetime. These mathematical entities describe the geometry of spacetime and are necessary to account for the influence of gravity on electromagnetic calculations.

The metric tensor $g_{\mu\nu}$ in Kerr space, expressed in Boyer-Lindquist coordinates, describes the geometry around a rotating black hole. It is given as follows:

$$g_{tt} = -\left(1 - \frac{2Mr}{\Sigma}\right), \tag{13}$$

$$g_{t\phi} = -\frac{2Mar\sin^2\theta}{\Sigma}, \tag{14}$$

$$g_{rr} = \frac{\Sigma}{\Delta}, \tag{15}$$

$$g_{\theta\theta} = \Sigma, \tag{16}$$

$$g_{\phi\phi} = \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right)\sin^2\theta, \tag{17}$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$. All other components of $g_{\mu\nu}$ that are not specified are zero. This tensor is used to determine how spacetime is curved due to the mass and rotation of the black hole.

The Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ are derived from the metric tensor and play a key role in determining the covariant derivative in curved spacetime. They are defined by the following formula:

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2}g^{\lambda\sigma}\left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right), \tag{18}$$

where $g^{\lambda\sigma}$ is the inverse of the metric tensor, and ∂ denotes the partial derivative. The calculation of these symbols involves the derivatives of the components of the metric tensor, which are then used to calculate the covariant derivatives of the electromagnetic field tensor.

Both the metric tensor and the Christoffel symbols play an important role in the formulation of Maxwell's equations in curved spacetime [20]. Specifically:

- The metric tensor is used to raise and lower indices in the tensor equations and to calculate the inverse metric, which is necessary to define the Christoffel symbols.
- The Christoffel symbols enter into the calculation of covariant derivatives, which replace the usual derivatives in Maxwell's equations. This modification ensures that the laws of electromagnetism in curved spacetime are consistent and that the principles of general relativity are respected.

The accurate calculation of these quantities is fundamental to the study of electrodynamics near black holes, providing the necessary tools to understand how electromagnetic fields behave in the curved geometry of spacetime around these massive objects.

Estimation of Electric and Magnetic Fields

Estimation of electric and magnetic fields near a black hole in a curved spacetime is a complex problem requiring careful consideration of both the physical foundations and the computational approach. Based on the program structure described earlier, this section contains a methodology for calculating these fields, including the corresponding formulas and approximations.

Estimation of the E and B fields near a black hole involves solving Maxwell's equations in a curved spacetime described by the Kerr metric. The computational approach is based on the following steps [21]:

1. Calculation of the metric tensor and Christoffel symbols at each point of the discretized spacetime.

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2. Use of these quantities to numerically solve the modified Maxwell equations.

3. Application of appropriate initial and boundary conditions to accurately model the physical situation.

Maxwell's equations in a curvilinear spacetime can be expressed in terms of the electromagnetic field tensor $F^{\mu\nu}$ as follows:

$$\nabla_\mu F^{\mu\nu} = \mu_0 J^\nu, \tag{19}$$
$$\nabla_{[\alpha} F_{\beta\gamma]} = 0, \tag{20}$$

where ∇_μ denotes the covariant derivative and J^ν is the four-dimensional current density vector. The electric and magnetic fields are the components of the field tensor $F^{\mu\nu}$ that need to be found in a given spacetime geometry [22].

The fields at (r, θ) in curvilinear Boyer-Lindquist coordinates are estimated using the following simplified formulas, taking into account axial symmetry and ignoring time dependence:

$$E_r = \frac{\rho}{4\pi\epsilon_0 r^2}, \tag{21}$$
$$E_\theta = E_\phi = 0, \tag{22}$$
$$B_r = \frac{\mu_0 j_\theta}{2\pi r}, \tag{23}$$
$$B_\theta = -\frac{\mu_0 j_r}{2\pi r}, \tag{24}$$
$$B_\phi = 0. \tag{25}$$

These formulas are based on the following assumptions:

- The electric field is mainly radial and proportional to the local charge density ρ, following a simplified form of the law Gauss.
- The magnetic field components are estimated using the basic form of Ampère's law, linearly relating them to the current density components j_r and j_θ.
- The non-radial components of the electric field and the azimuthal component of the magnetic field are assumed to be negligible or zero for simplicity.

Appropriate boundary conditions are applied at the event horizon and at large distances from the black hole. These conditions are chosen to reflect the physical behavior of the fields under the influence of strong gravity, such as the absorption of electromagnetic radiation by the black hole and the asymptotic decay of the fields [23].

The combination of these computational strategies, formulas, and simplifications provides a practical framework for estimating the behavior of electromagnetic fields in the extreme environment around black holes. Although some nuances of the exact behavior of the fields may be lost, this approach offers a balance between computational feasibility and physical realism.

Step-by-Step Calculation

This section provides a detailed step-by-step description of the process of calculating the electric and magnetic fields at a specific point in space, described by the Boyer-Lindquist coordinates (r, θ), around a rotating black hole. The calculation follows the previously described computational approach and is applied to a specified set of physical parameters.

The calculation starts with initialization of the physical parameters at the given coordinate:

1. Definition of the coordinate point: (r, θ).

2. Specification of the charge density (ρ) and components of the current density (j_r, j_θ, j_ϕ) at the given point.

3. Initialization of the mass M and angular momentum J (or rotation parameter a) of the black hole.

Calculation of the metric tensor and Christoffel symbols:

1. Calculation of the components of the Kerr metric tensor $g_{\mu\nu}$ at the point (r, θ) using the given mass and angular momentum of the black hole.

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2. Calculate the inverse metric tensor $g^{\mu\nu}$, which is needed to calculate the Christoffel symbols.
3. Determine the Christoffel symbols $\Gamma^{\lambda}_{\mu\nu}$ at a point using the metric tensor and its partial derivatives.

Estimate the electric and magnetic fields:

1. Use the charge density ρ to estimate the components of the electric field:

$$E_r = \frac{\rho}{4\pi\epsilon_0 r^2}, \tag{26}$$

$$E_\theta = E_\phi = 0. \tag{27}$$

   This approximation assumes a predominantly radial electric field due to the spherical symmetry of the charge distribution.

2. Estimating magnetic field components using current density values:

$$B_r = \frac{\mu_0 j_\theta}{2\pi r}, \tag{28}$$

$$B_\theta = -\frac{\mu_0 j_r}{2\pi r}, \tag{29}$$

$$B_\phi = 0. \tag{30}$$

   This method follows a simplified form of Ampère's law, assuming a linear relationship between current density and magnetic field.

The equations used in this calculation are based on classical electromagnetic theory, adapted to the context of curved spacetime. The electric field estimate is based on a Gaussian-like approach, while the magnetic field estimate uses a simplified form of Ampère's law. These simplifications are necessary to make the problem tractable and provide an approximate representation of the first-order fields. However, they may not account for all the complexities of electromagnetic field behavior in the strong gravitational field of a black hole.

This step-by-step calculation provides a fundamental approach to understanding how electric and magnetic fields behave in the curved spacetime around a black hole, providing insight into the interaction of electromagnetism and gravity in these extreme environments.

V. SUMMARY

1. Coordinate System and Assumptions: The cylindrical coordinate system (r, , z) is established as the primary framework for analysis. Key assumptions include the presence of a uniform external electromagnetic field and a spherically symmetric compact gravitational object.
2. Lorentz Force and Motion: The motion of a charged particle is governed by the Lorentz force, expressed as:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

   Here, $\vec{E}$ and $\vec{B}$ represent the electric and magnetic fields, respectively, and q is the charge of the particle. The equations of motion are derived from Newton's second law, leading to a set of differential equations.
3. Gravitational Field Influence: The influence of the gravitational field is incorporated through the use of the Schwarzschild metric, which describes the spacetime geometry around a compact object. The geodesic equations are modified to include the effects of both gravity and electromagnetism.
4. Particle Trajectories: By solving the derived equations, one can analyze the trajectories of charged particles in the vicinity of a compact object. Special emphasis is placed on the stability of these trajectories and their dependence on the strength of the external electromagnetic field.
5. Applications and Implications: The results have significant implications for astrophysical phenomena such as accretion disks around black holes, cosmic rays in magnetic fields, and plasma behavior in astrophysical jets. Understanding these dynamics contributes to our knowledge of particle behavior in extreme environments and the fundamental interactions governing such systems.

In conclusion, this review consolidates foundational ideas related to the motion of particles in combined gravitational and electromagnetic fields in a cylindrical coordinate system. Future studies could expand on these concepts, exploring more complex scenarios and the incorporation of additional physical effects.

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