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Monopole and Quadrupole Collective Modes of Anyons Confined in an Anisotropic Harmonic Potential

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Abstract. This study investigates the impact of a weak, three-dimensional ($3D$) isotropic laser speckle disorder on the thermodynamic properties of a dilute Bose gas at zero temperature. Initially, we revisit the derivation of the autocorrelation function for laser speckles in $1D$ and $2D$, adhering to the framework established by Goodman. This analysis demonstrates that the Gaussian approximation for this function, employed in some recent studies, deviates from the fundamental principles of laser speckle theory. Furthermore, we establish that the notion of a quasi-three-dimensional speckle, arising from the extension of the autocorrelation function along the longitudinal direction of a transverse $2D$ speckle, is not applicable to a genuine $3D$ speckle, as it necessitates an additional spatial dimension. Subsequently, we propose a feasible experimental realization of an isotropic $3D$ laser speckle potential and determine its corresponding autocorrelation function. Employing the Fourier transform of this function, we compute both the condensate depletion and the sound velocity of a Bose-Einstein condensate. These calculations are performed by averaging over the disorder ensemble for a weak laser speckle potential, within the framework of a perturbative solution to the Gross-Pitaevskii equation. This approach allows us to recover the expression for the normal fluid density previously derived via Landau's formalism. This physically intuitive derivation elucidates that the condensate particles scattered by the disorder constitute a gas of quasiparticles, giving rise to the normal fluid component.

Key words: Gross-Pitaevskii equation,isotropic laser speckle, Fourier transformation, Bose-Einstein condensate, harmonic potential

I. INTRODUCTION

We study the collective motion of anyons in a 2D harmonic trap. Motivated by the realization of ultracold atomic gases and their potential to simulate diverse quantum phenomena, we focus on anyons due to their exotic exchange statistics and relevance to fractional quantum Hall states and high-temperature superconductivity.

We begin by reviewing the progress in understanding the collective modes of Bose-Einstein condensates (BECs) in traps, highlighting key theoretical approaches such as the hydrodynamic approximation [3], sum-rule approach [3], time-dependent variational methods [4], and scaling ansatz [5–7]. These theoretical predictions have been confirmed by experiments probing collective excitations at both zero [8] and finite temperatures [9]. Furthermore, we acknowledge the significant advancements in the study of trapped Fermi gases, particularly in achieving degenerate regimes [10] and analyzing their thermodynamic properties [11, 12] and collective modes in both normal [18–20] and superfluid phases [21, 22]. The recent experimental realization of lower-dimensional BECs [29, 30] through dimensional reduction techniques further motivates our focus on 2D systems.

This paper is structured as follows. Sec. II presents the fundamentals of laser speckle theory. In Sec. III, we derive the autocorrelation function and its Fourier transform for laser speckles in 1D to 3D, following the methodology outlined in Refs. [13, 14], with an emphasis on both isotropic and anisotropic cases. Section IV details a proposed experimental scheme for generating a 3D isotropic speckle pattern. It is crucial to note that we consider a true 3D speckle with comparable grain sizes in all directions, as opposed to the quasi-3D speckle with an extended autocorrelation function discussed in Ref. [15] and section 4.4.3 of Ref. [14]. This distinction is essential, as the quasi-3D approach relies on an additional spatial dimension and is only applicable to 1D or 2D speckles. Our focus on true 3D speckles aligns with the simulations performed in Refs. [16, 17]. In

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Sec. V, we analyze the influence of a weak 3D isotropic speckle potential on the thermodynamic properties of a dilute Bose gas at zero temperature. We employ a perturbative solution to the Gross-Pitaevskii equation to calculate the condensate depletion and sound velocity. Section VI demonstrates the derivation of the normal fluid density, originally obtained using Landau's formalism, within our framework. This derivation highlights the emergence of quasiparticles from the scattering of condensate particles by the disorder potential, contributing to the normal fluid component. Finally, Sec. VII and Sec. VIII provide a comprehensive summary and analysis of our results.

II. HAMILTONIAN OF ANYONS TRAPPED IN 2D ANISOTROPIC HARMONIC POTENTIAL

In this section we describe the Hamiltonian of anyons, localized in the 2D anisotropic trap, which expression will be taken from the paper [31] and, following to the paper of Ghost and Sinha [32], we write this system Langrangian.

The Hamiltonian of the gas of N anyons with mass m and charge e, confined in 2D parabolic well, is:

$$\hat{H} = \frac{1}{2m}\sum_{k=1}^{N}\left(\vec{p_k} + A_\nu\left(\vec{r_k}\right)\right)^2 + \sum_{k=1}^{N}\frac{m}{2}\left(\omega_x^2 x_k^2 + \omega_y^2 y_k^2\right). \tag{1}$$

Here $\vec{r_k}$ and $\vec{p_k}$ represent the position and momentum operators of the kth anyon in 2D space dimension,

$$A_\nu(\vec{r_k}) = \hbar\nu\sum_{j\neq k}\frac{\vec{e_z}\times\vec{r_{kj}}}{|\vec{r_{kj}}|^2} \tag{2}$$

is the anyon gauge vector potential [33], $\vec{r_{kj}} = \vec{r_k} - \vec{r_j}$ and $\vec{e_z}$ is the unit vector normal to the 2D plane. In the expression for vector potential $A_\nu(\vec{r_k})$, ν is the anyon factor and hereafter we assume that $0 \leq \nu \leq 1$, which means the variation of the anyon factor between bosonic and fermionic limits of anyons.

Our interest is the solution of the Schrödinger equation

$$i\hbar\frac{\partial\psi(\vec{R},t)}{\partial t} = \hat{H}\psi(\vec{R},T). \tag{3}$$

Let us consider first the term in the Hamiltonian $\hat{H}$, containing only the anyon vector potential $A_\nu(\vec{r_k})$. In the bosonic representation of anyons we take the system wave function in the form [34, 35]:

$$\psi\left(\vec{R},t\right) = \prod_{k=j} r_{kj}^{\nu}\Psi_T\left(\vec{R},t\right). \tag{4}$$

Here and above $\vec{R} = \{\vec{r_1},\vec{r_2},...,\vec{r_n}\}$ is the configuration space of the N anyons. The product on the right-hand side of this equation is the Jastrow-type wave function. It describes the short distance correlations between two particles due to anyonic (fermionic) statistics interaction.

By substituting the wave function of this form into Schrödinger equation (3) without the harmonic potential term, we obtain the equation:

$$i\hbar\frac{\partial\Psi_T\left(\vec{R},t\right)}{\partial t} = \left(\hat{H}_1 + \hat{H}_2\right)\Psi_T(\vec{R},t), \tag{5}$$

where

$$\hat{H}_1 = \sum_{k=1}^{N}\left(-\frac{\hbar^2\Delta_k}{2m} - \frac{\hbar^2\nu}{m}\sum_{j\neq k}\frac{\vec{r}_{kj}\cdot\vec{\nabla}_k}{|\vec{r}_{kj}|^2}\right) \tag{6}$$

and

$$\hat{H}_2 = -i\frac{\hbar}{m}\sum_{k=1}^{N}\left(\vec{A}_\nu(\vec{r_k})\cdot\vec{\nabla}_k + \nu\sum_{j\neq k}\frac{\vec{A}_\nu(\vec{r_k})\cdot\vec{r}_{kj}}{|\vec{r}_{kj}|^2}\right). \tag{7}$$

As in the paper [32] of Ghosh and Sinha, by introducing

$$a_0 = \sqrt{\frac{\hbar}{m\omega_0}},$$

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where $\omega_0 = \sqrt{\omega_x \omega_y}$, we make dimensionless the length quantities and denote them by tilde sign.

We express the energy quantities in the Hamiltonian (1) in the units of $\hbar\omega_0$. Then, for instance, the harmonic potential term will have the form:

$$\frac{m}{2}\sum_{k=1}^{N}\left(\omega_x^2 x_k^2 + \omega_y^2 y_k^2\right) = \frac{\hbar\omega_0}{2}\sum_{k=1}^{N}\left(\lambda \tilde{x_k}^2 + \frac{1}{\lambda}\tilde{y_k}^2\right), \tag{8}$$

where $\lambda = \omega_x/\omega_y$, $\tilde{x}_k = x_k/a_0$, $\tilde{y}_k = y_k/a_0$ and parameter λ is the anisotropic parameter for the harmonic potential.

Now we make dimensionless Hamiltonians $\hat{H}_1$ and $\hat{H}_2$

$$\hat{\tilde{H}}_1 = \sum_{k=1}^{N}\left(-\frac{\hbar^2\Delta_k}{2m} - \frac{\hbar^2\nu}{m}\sum_{j\neq k}\frac{\vec{r}_{kj}\cdot\vec{\nabla}_k}{|\vec{r}_{kj}|^2}\right) = \tag{9}$$

$$-\hbar\omega_0\sum_{k=1}^{N}\left(\frac{\tilde{\Delta}_k}{2} + \nu\sum_{j\neq k}\frac{\tilde{\vec{r}}_{kj}\cdot\tilde{\vec{\nabla}}_k}{|\tilde{\vec{r}}_{kj}|^2}\right)$$

since $\Delta_k = \partial^2/\partial x_k^2 + \partial^2/\partial y_k^2 = \tilde{\Delta}_k/a_0^2$.

Similarly

$$\hat{\tilde{H}}_2 = -i\hbar\omega_0\sum_{k=1}^{N}\left(\tilde{\vec{A}}_\nu(\tilde{\vec{r}}_k)\cdot\tilde{\vec{\nabla}}_k + \nu\sum_{j\neq k}\frac{\tilde{\vec{A}}_\nu(\tilde{\vec{r}}_k)\cdot\tilde{\vec{r}}_{kj}}{|\tilde{\vec{r}}_{kj}|^2}\right), \tag{10}$$

where

$$\tilde{\vec{A}}_\nu(\tilde{\vec{r}}_k) = \nu\sum_{j\neq k}\frac{\vec{e}_z\times\tilde{\vec{r}}_{kj}}{|\tilde{\vec{r}}_{kj}|^2}$$

And finally, we obtain the dimensionless Schrödinger equation:

$$i\frac{\partial\Psi_T\left(\tilde{\vec{R}},\tilde{t}\right)}{\partial\tilde{t}} = \left(\hat{\tilde{H}}_1 + \hat{\tilde{H}}_2 + \frac{1}{2}\sum_{k=1}^{N}\left(\lambda\tilde{x_k}^2 + \frac{1}{\lambda}\tilde{y_k}^2\right)\right)\Psi_T(\tilde{\vec{R}},\tilde{t})$$

with $\tilde{t} = \omega_0 t$.

At the end of this section, we emphasize that the wave function $\Psi_T(\tilde{\vec{R}},\tilde{t})$ contains the configurational space of N anyons vector $\vec{R}$. Therefore, it corresponds to the many-particle wave function of the system. Previously, at the calculation of time variation of BEC, the wave function was a function of only one coordinate of condensate (see, for example, the paper [27]) and the solution of the problem of BEC collective motions in the harmonic trap was essentially easier.

III. LANGRANGIAN DENSITY OF ANYON GAS IN 2D ANISOTROPIC HARMONIC POTENTIAL

By taking into account the mentioned in the previous section, the Hamiltonian of anyons in the 2D harmonic potential, one can write down the Lagrangian density corresponding to our system. Throughout below, we assume that all quantities are dimensionless, which means that $\tilde{t} = \omega_0 t$, $\tilde{\vec{r}} = \vec{r}/a_0$ and energy quantities are expressed, as we said before, in $\hbar\omega_0$ units, and, for the simplicity, we omit everywhere the tilde sign.

Then the expression for the Lagrangian density is:

$$\mathcal{L} = \frac{i}{2}\left(\Psi_T\frac{\partial\Psi_T^*}{\partial t} - \Psi_T^*\frac{\partial\Psi_T}{\partial t}\right) -$$

$$\sum_{k=1}^{N}\left(\frac{1}{2}|\vec{\nabla}_k\Psi_T|^2 + \nu\Psi_T^*\sum_{j\neq k}\frac{\vec{r}_{kj}\cdot\vec{\nabla}_k}{|\vec{r}_{kj}|^2}\Psi_T\right) -$$

$$i\Psi_T^*\sum_{k=1}^{N}\left(\vec{A}_\nu(\vec{r}_k)\cdot\vec{\nabla}_k + \nu\sum_{j\neq k}\frac{\vec{A}_\nu(\vec{r}_k)\cdot\vec{r}_{kj}}{|\vec{r}_{kj}|^2}\right)\Psi_T +$$

$$\frac{1}{2}\sum_{k=1}^{N}\left(\lambda x_k^2 + \frac{1}{\lambda}y_k^2\right)|\Psi_T|^2. \tag{11}$$

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We take the trial wave function $\Psi_T(\vec{R},t)$ in the form:

$$\Psi_T(\vec{R},t) = C(t)\prod_{k=1}^{N}\exp\left[-\frac{1}{2}\left(\alpha(t)x_k^2+\beta(t)y_k^2+\gamma(t)x_k y_k\right)\right], \tag{12}$$

where it was assumed that the quantities $\alpha=\alpha_1+i\alpha_2$, $\beta=\beta_1+i\beta_2$ and $\gamma=\gamma_1+i\gamma_2$ are time dependent dimensionless complex variational parameters. This general form of the variational wave function includes the consideration of a monopole (dipole) collective mode motion, a quadrupole collective mode one, and also a scissor collective mode one. In the following, we omit the consideration of the scissor mode, which is a result of the last term in the expression for the wave function $\Psi_T(\vec{R},t)$, and consider only two former ones. However, in this section, by taking into account the calculation in the future the scissor mode frequency, we preserve all terms in the Lagrangian, responsible for this mode.

The Gaussian variational ansatz has a general form. It can be applied for the calculation of the ground state energy of the system when time-dependent parts of parameters α, β and γ are omitted. For the consideration of monopole, quadrupole and scissor oscillations, we use the most general time-dependent quadratic exponent of this variational ansatz. As seen in the Fig. 4.1 of the paper [36], physically, the monopole (dipole) breathing oscillation corresponds to the equal and simultaneous oscillation of x and y components of the system, while for the quadrupole oscillation, x and y components of the system move in the antiphase, i.e. if x goes to a minimum, then y goes to maximum and vice versa. For the scissor mode, there is an oscillation of the angle of any diagonal of the system with respect to the center of the coordinate.

We start with calculation of the normalization constant $C(t)$ for the above general form of $\Psi_T(\vec{R},t)$, using the normalization condition. However, before, after integration over dX and dY, we have

$$\int\int|\Psi_T|^2dXdY = |C(t)|^2\times \tag{13}$$
$$\prod_{k=1}^{N}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\exp\left[-\left(\alpha_1x_k^2+\beta_1y_k^2+\gamma_1x_ky_k\right)\right]dx_kdy_k$$

At the calculation of this integral, we use the formula 3.323.2 of Ref. [37]. This formula reads as

$$\int_{-\infty}^{+\infty}\exp\left(-px^2\pm qx\right) = \exp\left(\frac{q^2}{4p^2}\right)\frac{\sqrt{\pi}}{p},(p>0) \tag{14}$$

Making use of that formula and the normalization condition, one obtains:

$$|C(t)|^2 = \left(\frac{\sqrt{\alpha_1\beta_1-\gamma_1^2/4}}{\pi}\right)^N. \tag{15}$$

Since the coefficient $|C(t)|^2$ is real, it depends from real parts α_1, β_1, and γ_1 of variational parameters α, β, and γ.

Our goal is to calculate the effective Lagrangian L, integrating the Lagrangian density $\mathcal{L}$ over the 2D space co-ordinates, i.e. to get the expression

$$\frac{L}{N\hbar\omega_0} = \int\mathcal{L}\prod_{k=1}^{N}dx_kdy_k. \tag{16}$$

Since the Lagrangian density $\mathcal{L}$, Eq. (11), contains two parts: the one for ideal gas of particles in the harmonic potential and another one, who contains the anyon terms. The first part, we write as $\mathcal{L}_1+\mathcal{L}_2$ with

$$\mathcal{L}_1 = \frac{i}{2}\left(\Psi_T\frac{\partial\Psi_T^*}{\partial t}-\Psi_T^*\frac{\partial\Psi_T}{\partial t}\right)+\frac{1}{2}\sum_{k=1}^{N}\left(\lambda x_k^2+\frac{1}{\lambda}y_k^2\right)|\Psi_T|^2$$

and

$$\mathcal{L}_2 = \sum_{k=1}^{N}\frac{1}{2}|\vec{\nabla}_k\Psi_T|^2. \tag{17}$$

The second anyon part of $\mathcal{L}$ contains the terms:

$$\mathcal{L}_3 = \sum_{k=1}^{N}\nu\Psi_T^*\sum_{j\neq k}\frac{\vec{r}_{kj}\cdot\vec{\nabla}_k}{|\vec{r}_{kj}|^2}\Psi_T, \tag{18}$$

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$$\mathcal{L}_4 = -i\sum_{k=1}^{N}\Psi_T^*\sum_{k=1}^{N}\vec{A}_\nu(\vec{r}_k)\cdot\vec{\nabla}_k\Psi_T, \tag{19}$$

and

$$\mathcal{L}_5 = -i\sum_{k=1}^{N}\Psi_T^*\nu\sum_{j\neq k}\frac{\vec{A}_\nu(\vec{r}_k)\cdot\vec{r}_{kj}}{|\vec{r}_{kj}|^2}\Psi_T. \tag{20}$$

IV. EFFECTIVE LANGRANGIAN OF AN IDEAL GAS OF PARTICLES IN 2D ANISOTROPIC HARMONIC POTENTIAL

In this section, our goal is to calculate the effective Lagrangian L, Eq. (16), more exactly, to calculate parts of the L, related to ones of the Lagrangian density $\mathcal{L}$.

As we said before, first, we calculate the effective Lagrangian density of the ideal gas of particles in the 2D harmonic potential. We substitute the wave function $\Psi_T(\vec{R},t)$, Eq. (12), in the Lagrangian densities $\mathcal{L}_1$ and $\mathcal{L}_2$. By taking into account that

$$\begin{aligned}&\Psi_T\frac{\partial\Psi_T^*}{\partial t}-\Psi_T^*\frac{\partial\Psi_T}{\partial t}= \\ &i\sum_{k=1}^{N}\Big[\Big(\dot{\alpha}_2(t)x_k^2+\dot{\beta}_2(t)y_k^2+\dot{\gamma}_2(t)x_ky_k\Big)\times \\ &\exp\left[-\left(\alpha_1(t)x_k^2+\beta_1(t)y_k^2+\gamma_1(t)x_ky_k\right)\right]\Big]\times \\ &\prod_{j\neq k}\exp\left[\alpha_1(t)x_j^2+\beta_1(t)y_j^2+\gamma_1(t)x_jy_j\right]|C(t)|^2,\end{aligned} \tag{21}$$

and that integrals over dx_k and dy_k can be calculated separately, one obtains

$$\int\mathcal{L}_1\prod_{k=1}^{N}dx_kdy_k=\frac{1}{4\alpha_1\beta_1-\gamma_1^2}\times\left[-\left(\dot{\alpha}_2\beta_1+\dot{\beta}_2\alpha_1-2\gamma_1\dot{\gamma}_2\right)+\lambda\beta_1+\frac{\alpha_1}{\lambda}\right]$$

From here and throughout below all quantities of parameters α, β and γ are time dependent. The integration over dx_k and dy_k of $\mathcal{L}_2$ gives:

$$\begin{aligned}&\int\mathcal{L}_2\prod_{k=1}^{N}dx_kdy_k=\frac{1}{4\alpha_1\beta_1-\gamma_1^2} \\ &\times[\beta_1(\alpha_1^2+\alpha_2^2+\frac{1}{4}(\gamma_1^2+\gamma_2^2))+\alpha_1(\beta_1^2+\beta_2^2+\frac{1}{4}(\gamma_1^2+\gamma_2^2))-2\gamma_1(\alpha_1\gamma_1+\alpha_2\gamma_2+\beta_1\gamma_1+\beta_2\gamma_2)]\end{aligned} \tag{22}$$

With the last formula, we complete the expressions for effective Lagrangian of an ideal gas of particles without interaction, confined in 2D harmonic potential.

V. EFFECTIVE LANGRANGIAN OF ANYON GAS IN 2D ANISOTROPIC HARMONIC POTENTIAL

Now, we start to calculate the effective Lagrangian terms, including the anyon parts of the Lagrangian density. However, we restrict ourselves by consideration in the wave function $\Psi_T(\vec{R},t)$, Eq. (12), only the Gaussian wave function, i.e. omitting in the exponent of $\Psi_T(\vec{R},t)$ terms, proportional to γ. This means that, at the end, we consider only monopole and quadrupole collective modes.

We calculate the term in the effective Lagrangian related to

$$\mathcal{L}_3=\sum_{k=1}^{N}\nu\Psi_T^*\sum_{j\neq k}\frac{\vec{r}_{kj}\cdot\vec{\nabla}_k}{|\vec{r}_{kj}|^2}\Psi_T. \tag{23}$$

The simple calculation gives a formula:

$$\Psi_T^*\frac{\vec{r}_{kj}\cdot\vec{\nabla}_k}{|\vec{r}_{kj}|^2}\Psi_T=-\frac{\alpha_1^0x_kx_{kj}+\beta_1^0y_ky_{kj}}{x_{kj}^2+y_{kj}^2}|\Psi_T|^2, \tag{24}$$

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where $\alpha_1^0 = \alpha_1 + i\alpha_2$, $\beta_1^0 = \beta_1 + i\beta_2$, $x_{kj} = x_k - x_j$ and $y_{kj} = y_k - y_j$.

We introduce the function

$$A = \exp[-(\alpha_1 x_k^2 + \beta_1 y_k^2) - (\alpha_1 x_j^2 + \beta_1 y_j^2)] = \exp[-\alpha_1 x_{kj}^2 + 2\alpha_1 x_k x_{kj} - 2\alpha_1 x_k^2] \times \exp[-\beta_1 y_{kj}^2 + 2\beta_1 y_k y_{kj} - 2\beta_1 y_k^2]. \quad (25)$$

Our goal is to calculate the integral:

$$\int\int\int\int dx_k dy_k dx_j dy_j \frac{x_k x_{kj}}{x_{kj}^2 + y_{kj}^2} A = \int\int\int\int dx_k dy_k dx_{kj} dy_{kj} \frac{x_k x_{kj}}{x_{kj}^2 + y_{kj}^2} A, \quad (26)$$

where the lower and upper limits of all integrals are minus infinity and plus infinity, respectively.

We use the formula 3.462.2 of Ref. [37]

$$\int_{-\infty}^{+\infty} x^n e^{-px^2+2qx} dx = n! e^{\frac{q^2}{p}} \sqrt{\frac{\pi}{p}} \left(\frac{q}{p}\right)^n \sum_{k=0}^{E\left(\frac{n}{2}\right)} \frac{1}{(n-2k)!(k)!} \left(\frac{p}{4q^2}\right)^k,$$

where n is a positive integer, $E(n/2)$ is an integer part of argument $n/2$ and it should be $p > 0$, to calculate first integral over dy_k and then over dx_k. We obtain

$$\int\int\int\int dx_k dy_k dx_{kj} dy_{kj} \frac{x_k x_{kj} A}{x_{kj}^2 + y_{kj}^2} = \frac{\pi}{4\sqrt{\alpha_1 \beta_1}} \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_{kj} dy_{kj} \frac{x_{kj}^2}{x_{kj}^2 + y_{kj}^2} e^{-\frac{\alpha_1}{2} x_{kj}^2 - \frac{\beta_1}{2} y_{kj}^2}. \quad (27)$$

A similar calculation provides the formula:

$$\int\int\int\int dx_k dy_k dx_{kj} dy_{kj} \frac{y_k y_{kj} A}{x_{kj}^2 + y_{kj}^2} = \frac{\pi}{4\sqrt{\alpha_1 \beta_1}} \times \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_{kj} dy_{kj} \frac{y_{kj}^2}{x_{kj}^2 + y_{kj}^2} e^{-\frac{\alpha_1}{2} x_{kj}^2 - \frac{\beta_1}{2} y_{kj}^2}. \quad (28)$$

Therefore, we obtain the integral:

$$-\int\int\int\int dx_k dy_k dx_{kj} dy_{kj} \frac{\alpha_1^0 x_k x_{kj} + \beta_1^0 y_k y_{kj}}{x_{kj}^2 + y_{kj}^2} A = -\frac{\pi}{4} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_{kj} dy_{kj} \frac{\frac{\alpha_1^0}{\sqrt{\alpha_1 \beta_1}} x_{kj}^2 + \frac{\beta_1^0}{\sqrt{\alpha_1 \beta_1}} y_{kj}^2}{x_{kj}^2 + y_{kj}^2} \times e^{-\frac{\alpha_1}{2} x_{kj}^2 - \frac{\beta_1}{2} y_{kj}^2}. \quad (29)$$

Now, we have to calculate the integral:

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} dx_{kj} dy_{kj} \frac{x_{kj}^2}{x_{kj}^2 + y_{kj}^2} e^{-\frac{\alpha_1}{2} x_{kj}^2 - \frac{\beta_1}{2} y_{kj}^2}. \quad (30)$$

At the calculation of that integral over dx_{kj}, we use the formula 3.466.2 of the book [37]:

$$\int_0^{+\infty} dx \frac{x^2}{x^2 + \beta^2} e^{-\mu^2 x^2} = \frac{\pi}{2\mu} - \frac{\pi\beta}{2} e^{\mu^2 \beta^2} [1 - erf(\beta\mu)], \quad (31)$$

with $[Re\beta > 0, |arg\mu < \pi/4|]$ and $erf(x)$ is the error function.

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At the calculation of obtained integral over the dy_{kj}, we use the formula 6.289.2 of the same book [37] of I.S. Gradshteyn and I.M. Ryzhik:

$$\int_0^{+\infty} erf(\beta x)e^{(\beta^2-\mu^2)x^2}xdx = \frac{\beta}{2\mu(\mu^2-\beta^2)}, \tag{32}$$

at $[Re\mu^2 > Re\beta^2, |arg\mu < \pi/4|]$.

We obtain finally the expression

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx_{kj}dy_{kj}\frac{\frac{\alpha_1^0}{\sqrt{\alpha_1\beta_1}}x_{kj}^2+\frac{\beta_1^0}{\sqrt{\alpha_1\beta_1}}y_{kj}^2}{x_{kj}^2+y_{kj}^2}\times$$
$$e^{-\frac{\alpha_1}{2}x_{kj}^2-\frac{\beta_1}{2}y_{kj}^2} = i\frac{2\pi}{\alpha_1\beta_1}\frac{\alpha_1\beta_2-\alpha_2\beta_1}{\alpha_1-\beta_1} \tag{33}$$

for $\alpha_1 > \beta_1$ and

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dx_{kj}dy_{kj}\frac{\frac{\alpha_1^0}{\sqrt{\alpha_1\beta_1}}x_{kj}^2+\frac{\beta_1^0}{\sqrt{\alpha_1\beta_1}}y_{kj}^2}{x_{kj}^2+y_{kj}^2}\times$$
$$e^{-\frac{\alpha_1}{2}x_{kj}^2-\frac{\beta_1}{2}y_{kj}^2} = i\frac{2\pi}{\alpha_1\beta_1}\frac{\beta_1\alpha_2-\beta_2\alpha_1}{\beta_1-\alpha_1} \tag{34}$$

for $\beta_1 > \alpha_1$.

The first integral has an assymptotics: $-i2\pi\alpha_2/\alpha_1^2$, when $\beta_1 \to \alpha_1$ and $\beta_2 \to \alpha_2$ for $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$. The second integral has an asymptotics: $-i2\pi\beta_2/\beta_1^2$, when $\alpha_1 \to \beta_1$ and $\alpha_2 \to \beta_2$ for $\beta_1 > \alpha_1$ and $\beta_2 > \alpha_2$. This is an expression for isotropic 2D harmonic potential.

By taking into account the expression for constant $|C(t)|^2$, Eq. (15), with $\gamma_1 = 0$, for isotropic harmonic potential, when $\beta_1^0 = \alpha_1^0$, we have the expression for effective Lagrangian

$$\nu\int \mathcal{L}_3\prod_{k=1}^{N} dx_k dy_k = -N(N-1)\frac{\alpha_1^0}{2}\nu \tag{35}$$

For the general case of anisotropic 2D harmonic potential, when we may investigate the parameters β_1^0 not equal α_1^0, but $\beta_1^0 \approx \alpha_1^0$, the imaginary parts of expression (35) exactly cancel the asymptotics of the above first integral. We remind that in this case it were two conditions $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$.

One can write the analog of Eq. (35) for the second conditions $\beta_1 > \alpha_1$ and $\beta_2 > \alpha_2$. In this case

$$\nu\int \mathcal{L}_3\prod_{k=1}^{N} dx_k dy_k = -N(N-1)\frac{\beta_1^0}{2}\nu. \tag{36}$$

Again, for anisotropic 2D harmonic potential with $\beta_1 > \alpha_1$ and $\beta_2 > \alpha_2$ but for $\alpha_1^0 \approx \beta_1^0$ the imaginary parts of expression (36) exactly cancel the asymptotics of the above second integral.

One can write the symmetric form of this effective Lagrangian for the general case of isotropic and anisotropic harmonic potential in the form:

$$\nu\int \mathcal{L}_3\prod_{k=1}^{N} dx_k dy_k = -N(N-1)\frac{1}{2}\left(\frac{\alpha_1}{2}+\frac{\beta_1}{2}\right)\nu. \tag{37}$$

It is interesting to note that the last expression for the effective Lagrangian is valid in the first order on small parameters $\delta = \alpha_1 - \beta_1$ and $\delta = \alpha_2 - \beta_2$ or $\delta = \beta_1 - \alpha_1$ and $\delta = \beta_2 - \alpha_2$.

Now, we calculate the effective Lagrangian related to the Lagrangian density:

$$\mathcal{L}_4 = -i\sum_{k=1}^{N}\Psi_T^*\sum_{k=1}^{N}\vec{A}_\nu(\vec{r}_k)\cdot\vec{\nabla}_k\Psi_T. \tag{38}$$

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After substituting the expression for $\vec{A}_\nu(\vec{r}_k)$

$$\vec{A}_\nu(\vec{r}_k) = \nu \sum_{j\neq k} \frac{-y_{kj}\vec{i} + x_{kj}\vec{j}}{|\vec{r}_{kj}|^2} \tag{39}$$

in this formula for $\mathcal{L}_4$, we obtain

$$\Psi_T^* \vec{A}_\nu(\vec{r}_k) \cdot \vec{\nabla}_k \Psi_T = \nu \frac{\alpha_1^0 x_k y_{kj} - \beta_1^0 y_k x_{kj}}{x_{kj}^2 + y_{kj}^2} |\Psi_T|^2. \tag{40}$$

Our goal to calculate the integral:

$$\int\int\int\int dx_k dy_k dx_{kj} dy_{kj} \frac{x_k y_{kj}}{x_{kj}^2 + y_{kj}^2} A, \tag{41}$$

which we do following the procedure, described above. The result of this calculation is:

$$\begin{aligned} &\int\int\int\int dx_k dy_k dx_{kj} dy_{kj} \frac{x_k y_{kj} A}{x_{kj}^2 + y_{kj}^2} = \frac{\pi}{4\sqrt{\alpha_1 \beta_1}} \times \\ &\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dx_{kj} dy_{kj}}{x_{kj}^2 + y_{kj}^2} x_{kj} y_{kj} e^{-\frac{\alpha_1}{2} x_{kj}^2 - \frac{\beta_1}{2} y_{kj}^2} = 0. \end{aligned} \tag{42}$$

Because the integrand is an odd function as on x_{kj} so and on y_{kj}.

The similar result we have for integral

$$\begin{aligned} &\int\int\int\int dx_k dy_k dx_{kj} dy_{kj} \frac{y_k x_{kj} A}{x_{kj}^2 + y_{kj}^2} = \frac{\pi}{4\sqrt{\alpha_1 \beta_1}} \times \\ &\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{dx_{kj} dy_{kj}}{x_{kj}^2 + y_{kj}^2} y_{kj} x_{kj} e^{-\frac{\alpha_1}{2} x_{kj}^2 - \frac{\beta_1}{2} y_{kj}^2} = 0. \end{aligned} \tag{43}$$

This calculation shows that

$$\int \mathcal{L}_4 \prod_{k=1}^{N} dx_k dy_k = 0. \tag{44}$$

The last anyon part of the effective Lagrangian related to the Lagrangian density:

$$\mathcal{L}_5 = \sum_{k=1}^{N} \nu \sum_{j\neq k} \frac{\vec{A_\nu}(\vec{r_k}) \vec{r_{kj}}}{|\vec{r_{kj}}|^2} |\Psi_T|^2. \tag{45}$$

In the general case of 2D anisotropic harmonic potential with $\alpha \neq \beta$, it is impossible to calculate the space integral of the Lagrangian density $\mathcal{L}_5$. This is related with the impossibility to calculate integrals like

$$\int_{-\infty}^{+\infty} dx_{kl} \frac{1}{x_{kl}^2 + y_{kl}^2} e^{-\frac{2\alpha_1}{3} x_{kl}^2 + \frac{2\alpha_1}{3} x_{kj} x_{kl}}$$

or

$$\int_{-\infty}^{+\infty} dy_{kj} \frac{1}{x_{kj}^2 + y_{kj}^2} e^{-\frac{2\beta_1}{3} y_{kj}^2 + \frac{2\beta_1}{3} y_{kj} y_{kl}}.$$

However, for the isotropic case of the 2D harmonic potential with $\alpha = \beta$, one can exactly get the result:

$$\int \mathcal{L}_5 \prod_{k=1}^{N} dx_k dy_k = 0. \tag{46}$$

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Now we collect all terms of Lagrange function L and have

$$\frac{L}{N\hbar\omega_0} = \int \mathcal{L} \prod_{k=1}^{N} dx_k dy_k = \tag{47}$$
$$\int \mathcal{L}_1 \prod_{k=1}^{N} dx_k dy_k - \int \mathcal{L}_2 \prod_{k=1}^{N} dx_k dy_k - \int \mathcal{L}_3 \prod_{k=1}^{N} dx_k dy_k =$$
$$\frac{1}{4\alpha_1\beta_1}\left[-\left(\dot{\alpha_2}\beta_1 + \dot{\beta_2}\alpha_1\right) + \lambda\beta_1 + \frac{\alpha_1}{\lambda}\right] + \frac{1}{4\alpha_1\beta_1}\left[\beta_1\left(\alpha_1^2+\alpha_2^2\right) + \alpha_1\left(\beta_1^2+\beta_2^2\right)\right] + (N-1)\frac{\alpha_1+\beta_1}{4}\nu.$$

This expression for $\frac{L}{N\hbar\omega_0}$ is also valid for anisotropic case $\alpha_1 \neq \beta_1$ but for $\alpha_1 \approx \beta_1$. Since, everywhere above we calculated every equation for the general case $\alpha_1 \neq \beta_1$ with the exception integration of expression for $\mathcal{L}_5$.

VI. STATIC PROPERTIES OF ANYONS: THE GROUND STATE ENERGY

Now, we calculate the ground state energy of the anyon system with N particles, localized in the 2D anisotropic harmonic potential, at equilibrium. For that, we put $\alpha_2 = 0$, $\beta_2 = 0$ and $\gamma = 0$ in the formulas $\alpha = \alpha_1 + i\alpha_2$ and $\beta = \beta_1 + i\beta_2$ in the wave function expression (12). According to the definition, which connects the Lagrange function with the system stationary (ground) state energy, the expression for that energy is:

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = E = const. \tag{48}$$

Therefore,

$$\sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L = -\frac{1}{4}\left(\frac{\lambda}{\alpha_1} + \frac{1}{\lambda\beta_1}\right) - \frac{1}{4}(\alpha_1+\beta_1) -$$
$$\frac{(N-1)\nu}{4}(\alpha_1+\beta_1). \tag{49}$$

By changing $const \to -const$, one obtains the expression for energy:

$$E = \frac{1}{4}\left(\frac{\lambda}{\alpha_1} + \frac{1}{\lambda\beta_1}\right) + \frac{1}{4}(\alpha_1+\beta_1) + \frac{(N-1)\nu}{4}(\alpha_1+\beta_1). \tag{50}$$

To get the minimum of this energy E, one needs to minimize it with respect to variational parameters α_1 and β_1. The solutions of the couple of equations

$$\frac{dE}{d\alpha_1} = \frac{1}{4}\left(-\frac{\lambda}{\alpha_1^2} + 1 + \nu(N-1)\right) = 0$$

$$\frac{dE}{d\beta_1} = \frac{1}{4}\left(-\frac{1}{\lambda\beta_1^2} + 1 + \nu(N-1)\right) = 0$$

are:

$$\alpha_{10}^2 = \frac{\lambda}{(1+\nu(N-1))} \tag{51}$$

and

$$\beta_{10}^2 = \frac{1}{\lambda(1+\nu(N-1))}. \tag{52}$$

These parameters α_{10} and β_{10} provide the minimum energy E. To get it, we substitute the above-obtained expressions for α_{10} and β_{10} in equation for E, Eq. (50), and obtain the expression for the ground state energy:

$$\frac{E_0}{N\hbar\omega_0} = \frac{1}{2}\left(\sqrt{\lambda} + \frac{1}{\sqrt{\lambda}}\right)(1+\nu(N-1))^{\frac{1}{2}}, \tag{53}$$

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where

$$\omega_0 = \sqrt{\omega_x \omega_y}$$

and

$$\lambda = \frac{\omega_x}{\omega_y}.$$

For the case of $\lambda = 1$, $\omega_x = \omega_y$, we reproduce the ground state energy (see our paper [38]):

$$E_0 = \hbar\omega_k N \left(1 + \nu(N-1)\right)^{\frac{1}{2}}, \tag{54}$$

of N anyons, localized in the 2D isotropic harmonic potential. As we demonstrated in this paper, in the case of anyon parameter $\nu = 1$, when anyons reproduce fermions, the ground state energy with the high accuracy (less than 6 percent at big values of N) represents the exact one for each fixed N.

The expression for E_0, Eq. (53), allows also us to reproduce the ground state energy of anyons, localized in the one dimensional (1D) harmonic potential. Thus at

$$\omega_y = 0,$$

we derive

$$E_0 = \frac{\hbar\omega_x}{2} N \left(1 + \nu(N-1)\right)^{\frac{1}{2}}. \tag{55}$$

Again at $\nu = 1$, the ground state energies of fermions in 1D harmonic potential are well reproduced.

VII. COLLECTIVE MODES

In this section, we will derive collective mode frequencies. As was said before, we consider the monopole and quadrupole oscillations of the system of anyons in 2D harmonic potential.

It is convenient to use the system effective Lagrangian in the form:

$$\frac{L}{N\hbar\omega_0} = \frac{1}{4}\left[-\frac{\dot{\alpha}_2}{\alpha_1} - \frac{\dot{\beta}_2}{\beta_1} + \frac{\lambda}{\alpha_1} + \frac{1}{\lambda\beta_1} + \right.$$
$$\left. \alpha_1(1 + \nu(N-1)) + \beta_1(1 + \nu(N-1))\right]. \tag{56}$$

For systems of Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_2} - \frac{\partial L}{\partial \alpha_2} = 0$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}_2} - \frac{\partial L}{\partial \beta_2} = 0 \tag{57}$$

and

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}_1} - \frac{\partial L}{\partial \alpha_1} = 0$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\beta}_1} - \frac{\partial L}{\partial \beta_1} = 0 \tag{58}$$

we obtain systems of equations:

$$\dot{\alpha}_1 - 2\alpha_1\alpha_2 = 0$$
$$\dot{\beta}_1 - 2\beta_1\beta_2 = 0 \tag{59}$$

and

$$\dot{\alpha}_2 - (\lambda + \alpha_2^2) + \alpha_1^2(1 + \nu(N-1)) = 0$$
$$\dot{\beta}_2 - (1/\lambda + \beta_2^2) + \beta_1^2(1 + \nu(N-1)) = 0. \tag{60}$$

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Following to paper [32] of Ghosh and S.Sinha, we assume that $\alpha_1(t) = \alpha_{10} + \delta\alpha_1(t)$, $\beta_1(t) = \beta_{10} + \delta\beta_1(t)$, $\alpha_2 = \delta\alpha_2(t)$, $\beta_2(t) = \delta\beta_2(t)$ and preserve in the last two systems of equations first order terms on $\delta\alpha_1$, $\delta\beta_1$, $\delta\alpha_2$, $\delta\beta_2$.

Then we obtain the systems of the following equations:

$$\begin{aligned} \delta\dot{\alpha}_1 - 2\alpha_{10}\delta\alpha_2 = 0 \\ \delta\dot{\beta}_1 - 2\beta_{10}\delta\beta_2 = 0 \end{aligned} \tag{61}$$

and

$$\begin{aligned} \delta\dot{\alpha}_2 + 2\alpha_{10}\delta\alpha_1\,(1 + \nu(N-1)) = 0 \\ \delta\dot{\beta}_2 + 2\beta_{10}\delta\beta_1\,(1 + \nu(N-1)) = 0. \end{aligned} \tag{62}$$

For the first system of equations, we have

$$\begin{aligned} \delta\ddot{\alpha}_1 - 2\alpha_{10}\delta\dot{\alpha}_2 = 0 \\ \delta\ddot{\beta}_1 - 2\beta_{10}\delta\dot{\beta}_2 = 0 \end{aligned} \tag{63}$$

and substituting here $\dot{\alpha}_2$ and $\dot{\beta}_2$, taken from the second system of equation, one derives:

$$\begin{aligned} \delta\ddot{\alpha}_1 + 4\alpha_{10}^2\delta\alpha_1(1 + \nu(N-1)) = 0 \\ \delta\ddot{\beta}_1 + 4\beta_{10}^2\delta\beta_1(1 + \nu(N-1)) = 0. \end{aligned} \tag{64}$$

Using here (51) and (52) for α_{10} and β_{10}, one obtains the system of equations in the following form:

$$\begin{aligned} \delta\ddot{\alpha}_1 + 4\lambda\delta\alpha_1 = 0 \\ \delta\ddot{\beta}_1 + \frac{4}{\lambda}\delta\beta_1 = 0. \end{aligned} \tag{65}$$

Substituting here $\delta\alpha_1 = e^{i\omega_1 t}$ and $\delta\beta_1 = e^{i\omega_1 t}$ we find solutions of the last system of equations:

$$\begin{aligned} \omega_1^2 = 4\lambda \\ \omega_2^2 = \frac{4}{\lambda} \end{aligned} \tag{66}$$

We analyse these solutions and obtain frequencies for both isotropic and anisotropic cases mediums:

For the isotropic case, $\lambda = 1$, $\omega_1 = \omega_2 = 2\omega_0$, which coincides with ω_+ for monopole mode, as well as, with quadrupole mode for the non-interacting case.

When focused on anisotropic case, $\lambda \neq 1$, anisotropic trap monopole mode is described by $\omega_1 = 2\omega_x$ and quadrupole one by $\omega_2 = 2\omega_y$. As we can see, the values of the frequencies do not depend on anyonic factor ν.

VIII. COMPARISON OF RESULTS FOR COLLECTIVE MODES WITH PREVIOUS CALCULATION

We can check the correctness of our results by comparing them with the same non-interacting bosons, given in the paper [32] of Ghosh and Sinha, namely, with equation (17) of this paper. The equation (17) of [32] has the form:

$$\frac{\omega_\pm^2}{\omega_0^2} = \frac{\lambda}{2}\cdot\frac{8+3P\eta}{2+P\eta} + \frac{1}{2\lambda}\cdot\frac{8\eta+3\eta}{2\eta+P} \pm \sqrt{\left(\frac{\lambda(8+3P\eta)}{2(2+P\eta)} + \frac{8\eta+3\eta}{2\lambda(2\eta+P)}\right)^2 + \left(\frac{P\lambda\eta^2}{2+P\eta}\right)^2}. \tag{67}$$

If there is no interaction between bosons, then one assumes in this paper that $P = 0$ and $\eta = 1/\lambda$ and thus

$$\begin{aligned} \frac{\omega_+^2}{\omega_0^2} = 4\lambda \\ \frac{\omega_-^2}{\omega_0^2} = \frac{4}{\lambda}, \end{aligned} \tag{68}$$

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from where for non-interacting bosons in an anisotropic trap, we have in our notion:

$$\begin{aligned}\frac{\omega_1^2}{\omega_0^2} &= \frac{\omega_+^2}{\omega_0^2} \\ \frac{\omega_2^2}{\omega_0^2} &= \frac{\omega_-^2}{\omega_0^2}.\end{aligned} \tag{69}$$

Besides, there is another method of checking the accuracy of results we obtained in (66). This can be done by comparing them with result-equation (44) of paper [32] for non-interacting fermions. According to paper [32], monopole and quadrupole frequencies for fermions in the 2D harmonic trap are described by equation (44), which has the following form:

$$\begin{aligned}\frac{\omega_\pm^2}{\omega_0^2} = \frac{3K_0+1}{2K_0}\left(\lambda+\frac{1}{\lambda}\right) \pm \\ \sqrt{\left(\frac{3K_0+1}{2K_0}\left(\lambda+\frac{1}{\lambda}\right)\right)^2+\left(1-\frac{1}{K_0}\right)^2},\end{aligned} \tag{70}$$

where $K_0 = 1 + g_2 m/(\pi\hbar^2)$ and g_2 is an interaction constant. When there is no interaction among fermions, $g_2 = 0$ and $K_0 = 1$, as a result, we again obtain (69).

This result indicates that monopole and quadrupole modes of the ideal gas of 2D quantum particles are independent of boson and fermion statistics. An interesting fact is that if in Ref. [32] authors used different treatments for its derivation for bosons and fermions, our anyon consideration allows us to get it within the one approach. Another interesting fact is that, as it is seen from formulas (67) and (70), the statistics of particles start to play a role just after the inclusion of an interparticle interaction.

The independence of monopole and quadrupole modes of 2D harmonically trapped ideal quantum gases from particle statistics is a nontrivial result. To be convinced that this result is true, we have decided to calculate frequencies of these modes within other cumulants equation of motion approach, which was developed in Ref. [39] (see also Ref. [36]).

IX. SUMMARY AND CONCLUSION

This investigation explored the collective behavior of anyons confined in a $2D$ anisotropic harmonic potential, focusing on their dynamics at ultracold temperatures ($T \approx 0$). We employed a time-variational approach, developed within this study, to derive the ground-state energy and collective mode frequencies of the anyon system. Our analysis revealed that the frequencies of both monopole and quadrupole oscillations are independent of the anyonic statistical parameter.

Crucially, we demonstrated consistency with existing theoretical predictions. In the limiting cases of bosonic and fermionic statistics, our calculated collective mode frequencies precisely match those of bosons and fermions in $2D$ anisotropic harmonic potentials at $T \approx 0$. This agreement strongly supports the validity and accuracy of our chosen methodology.

This work provides valuable insights into the collective dynamics of anyons, enhancing our understanding of their unique properties. The developed time-variational approach offers a powerful tool for investigating multiparticle quantum systems, especially those involving particles with non-trivial exchange statistics. Looking ahead, potential research avenues include generalizing this framework to explore anyons in diverse trapping potentials, analyzing the impact of finite temperature effects, and elucidating the role of anyonic statistics in the emergence of novel quantum phases. This aligns with the broader interest in utilizing anyons to address fundamental questions in theoretical physics and to potentially facilitate advancements in quantum computation, as highlighted in the introduction.

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