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Scalar perturbation due to geodesic motion in the Schwarzschild spacetime

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Abstract. In this paper, we have studied scalar perturbations caused by the geodesic motion of a massive particle orbiting a black hole in Schwarzschild spacetime. For simplicity, we assumed that the scalar field does not contribute to the geodesic motion. We briefly discussed particle motion in Schwarzschild spacetime. We derived the equation for the scalar field using the Klein-Gordon equation with an included source term. We obtained a fully analytical solution to the Klein-Gordon equation in Schwarzschild spacetime. We demonstrated that the solution to the angular equation can be expressed in terms of spherical harmonics, while the radial equation solution can be expressed in terms of the confluent Heun function.

Key words: Scalar field, geodesic equation, confluent Heun function, Kerr spacetime, black hole

I. INTRODUCTION AND SUMMARY

In the context of general relativity, a perturbation in the Schwarzschild spacetime refers to small deviations or disturbances from the idealized Schwarzschild solution, which describes the spacetime geometry around a non-rotating, spherically symmetric mass (such as a black hole or a massive object). Perturbations in the Schwarzschild spacetime can arise due to the presence of additional masses, gravitational waves, or other forms of energy-momentum that cause deviations from the ideal Schwarzschild solution [1]. Using the Newman-Penrose formalism [2], the perturbation in Kerr spacetime has been studied in [3–6]. These perturbations can be analyzed using techniques from perturbation theory in general relativity, which involve linearizing the Einstein field equations around the Schwarzschild and Kerr metrics and solving for the perturbed metric components. Perturbations in the Schwarzschild and Kerr spacetime can have important physical implications, such as the generation and propagation of gravitational waves, the dynamics of test particles and light rays, and the stability of black holes against perturbations. They are also relevant for understanding the behavior of astrophysical systems containing compact objects like black holes and neutron stars.

In the context of general relativity, the Schwarzschild spacetime describes the gravitational field outside a spherically symmetric, non-rotating mass, such as a black hole or a star. Scalar perturbations in the Schwarzschild spacetime can be analyzed to understand how small deviations from the background metric behave. Scalar perturbations typically refer to small deviations from the background metric (in this case, the Schwarzschild metric) that can be expressed as changes in a scalar field. These perturbations can be studied by considering linearized equations around the Schwarzschild solution [7, 8].

Once the linearized perturbation equations are obtained, one can study the behavior of scalar perturbations due to geodesic motion by solving these equations and analyzing the resulting solutions. This analysis can provide insights into how small disturbances propagate in the Schwarzschild spacetime and how they interact with the background gravitational field. The study of scalar perturbations in the Schwarzschild spacetime due to geodesic motion is a topic of interest in theoretical physics and general relativity, and it has applications in understanding gravitational waves, black hole physics, and cosmology. Detailed calculations and analyses of such perturbations require a deep understanding of general relativity and mathematical techniques for solving partial differential equations in curved spacetime [9–11].

The geodesic motion in the Schwarzschild spacetime refers to the motion of test particles (massive or massless) moving under the influence of gravity alone, following the natural paths dictated by the curvature of spacetime. To analyze scalar perturbations due to geodesic motion in the Schwarzschild spacetime, one typically starts with the linearized perturbation equations, which are

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derived by decomposing the metric tensor into a background Schwarzschild metric plus a small perturbation. These perturbation equations can be obtained by considering small deviations in the metric tensor components and linearizing the Einstein field equations [12–14].

II. SCALAR PERTURBATION DUE TO GEODESIC MOTION

The scalar perturbation due to the geodesic motion is quit common problem and considered by numerous authors [9, 10]. The field equation for the scalar field Φ is given as

$$\nabla_\mu \nabla^\mu \Phi = -4\pi g_s T \,, \tag{1}$$

where ∇_μ stands for covariant derivative from a scalar field, m_0 is the mass of particle, g_s is the coupling constant between massive field and scalar field and $T = T_{\mu\nu} g^{\mu\nu}$ is a source term regarded the trace of energy-momentum tensor that represents as follows [9]:

$$T_{\mu\nu} = \frac{m_0}{\sqrt{-g}} \int d\tau \delta^{(4)}[x - z(\tau)] u_\mu u_\nu \tag{2}$$

$$T = T_{\mu\nu} g^{\mu\nu} = -\frac{m_0 \delta(\mathbf{r} - \mathbf{r}_0)}{u^0 r^2 \sin\theta} \,. \tag{3}$$

Here g is the determinant of the metric tensor, τ is a proper time, $u^\mu = dx^\mu/d\tau$ is the four-velocity of massive particle normalized as $u_\mu u^\mu = -1$, and satisfied a geodesic equation $u^\mu \nabla_\mu u^\nu = 0$. In the Schwarzschild spacetime, of the mass parameter M,

$$g_{\mu\nu} = \begin{pmatrix} -f & 0 & 0 & 0 \\ 0 & \frac{1}{f} & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta \end{pmatrix} \,, \qquad f = 1 - \frac{2M}{r} \,, \tag{4}$$

geodesic equation can be easily integrated and the four velocity of test particle can be expressed as

$$u_\mu = \left\{ \frac{\mathcal{E}}{f}, \sqrt{\mathcal{E}^2 - f\left(\frac{\mathcal{K}}{r^2} + 1\right)}, \frac{1}{r^2}\sqrt{\mathcal{K} - \frac{\mathcal{L}^2}{\sin^2\theta}}, \frac{\mathcal{L}}{r^2 \sin^2\theta} \right\} \,, \tag{5}$$

where $\mathcal{E}$ and $\mathcal{L}$ are constants of motion regarded to the specific energy, specific angular momentum of test particle and $\mathcal{K}$ is the Carter constant. The angular velocity of particle measured by a distant observer yields

$$\Omega = \frac{u^\phi}{u^t} = \frac{\mathcal{L} f}{\mathcal{E} r^2 \sin^2\theta} \,. \tag{6}$$

For simplicity, assuming particle moves in circular orbits with four-velocity of $u_\mu = \{-\mathcal{E}, 0, 0, \mathcal{L}\}$. In the Keplerian orbit measurable quantities, namely, the specific energy $\mathcal{E}$, specific angular momentum $\mathcal{L}$ and angular frequency Ω of massive test particle in the Schwarzschild spacetime are found as

$$\mathcal{E} = \frac{f}{\sqrt{1 - \frac{3M}{r}}} \,, \qquad \mathcal{L} = \frac{\sqrt{Mr}}{\sqrt{1 - \frac{3M}{r}}} \,, \qquad \Omega = \sqrt{\frac{M}{r^3}} \,. \tag{7}$$

Using equations (2) together with (5) and (7), the trace of the energy-momentum tensor for test particle T reduces to

$$T = -\frac{m_0}{r^2}\sqrt{1 - \frac{3M}{r}} \delta(r - r_0)\delta(\cos\theta)\delta(\phi - \Omega t) \,. \tag{8}$$

Finally, taking into account all facts mentioned above, equation for the scalar field in the Schwarzschild spacetime can be rewritten as

$$\left[\partial_t^2 - \frac{f}{r^2}\frac{\partial}{\partial r}\left(r^2 f \frac{\partial}{\partial r}\right) - \frac{f}{r^2}\nabla_\Omega^2\right] \Phi(\mathbf{r}, t) = -4\pi m_0 g_s \frac{f}{r^2}\sqrt{1 - \frac{3M}{r}} \delta(r - r_0)\delta(\cos\theta)\delta(\phi - \Omega t) \,, \tag{9}$$

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where ∇^2_Ω is the angular part of the Laplacian operator defined as

$$\nabla^2_\Omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2} \,. \tag{10}$$

The solution of equation (9) can expanded in terms of the spherical harmonics in the form:

$$\Phi(\mathbf{r},t) = 2mg_s\sum_{\ell m}\frac{R_{\ell m}(r)}{r\sqrt{f}}Y^m_\ell(\theta,\phi-\Omega t) \,, \tag{11}$$

where $R_{\ell m}(r)$ is an unknown radial function, $Y^m_\ell(\theta,\phi)$ is the spherical harmonics satisfied the

$$\nabla^2_\Omega Y^m_\ell(\theta,\phi) = -\ell(\ell+1)Y^m_\ell(\theta,\phi) \,, \tag{12}$$

and normalized as

$$\int Y^m_\ell(\theta,\phi)Y^{*m'}_{\ell'}(\theta,\phi)d\Omega = \delta_{\ell\ell'}\delta_{mm'} \,. \tag{13}$$

Now using the a property of the delta function:

$$\delta(\phi-\Omega t) = \frac{1}{2\pi}\sum_m e^{im(\phi-\Omega t)} \,, \tag{14}$$

the equation (9) can be rewritten in a form:

$$\sum_{\ell m}\left[\frac{d^2}{dr^2} + \frac{M^2+(m\Omega r^2)^2-\ell(\ell+1)r^2f}{r^4f^2}\right]R_{\ell m}Y^m_\ell(\theta,\phi-\Omega t) = \sqrt{f\left(1-\frac{3M}{r}\right)}\frac{\delta(r-r_0)}{r}\delta(\cos\theta)\sum_m e^{im(\phi-\Omega t)} \,, \tag{15}$$

where Ω is the angular frequency of Keplerian orbit. In order to rid off the angular dependence of the last equation one can multiply complex conjugate of the spherical harmonics Y^{*m}_ℓ from both side of the equation (15) and using normalization relation (13), one obtains

$$\left[\frac{d^2}{dr^2} + \frac{M^2+(m\Omega r^2)^2-\ell(\ell+1)r^2f}{r^4f^2}\right]R_{\ell m} = Y^{*m}_\ell(\frac{\pi}{2},0)\frac{\delta(r-r_0)}{r}\sqrt{f\left(1-\frac{3M}{r}\right)} \,, \tag{16}$$

As one can see that above equation is independent of the angular coordinate but it depends on the radial coordinate only.

Let us introduce new function $U(x)$ with new radial coordinate in the form:

$$R_{\ell m}(x) = e^{\alpha x/2}x^{\beta/2}(x-1)^{\gamma/2}U_{\ell m}(x) \,, \qquad x = 1-\frac{r}{2M} \,, \tag{17}$$

Inserting it into equation (15), one can get well-known Heun equation in the form:

$$U''_{\ell m} + \left(\alpha+\frac{\beta}{x}+\frac{\gamma}{x-1}\right)U'_{\ell m} + \left(\frac{\sigma}{x}+\frac{\lambda-\sigma}{x-1}\right)U_{\ell m} = 0, \tag{18}$$

and the solution can be expressed in term of the Heun function in the form: $U_{\ell m}(x) = \text{HeunC}[\sigma,\lambda,\beta,\gamma,\alpha,x]$, where coefficients are

$$\sigma = \ell^2+\ell-4imM\Omega, \quad \lambda = -4imM\Omega, \quad \alpha\to-4imM\Omega, \quad \beta = 1+4imM\Omega, \quad \gamma = 1 \,, \tag{19}$$

$$\sigma = \ell^2+\ell-16m^2M^2\Omega^2, \quad \lambda = 4mM\Omega(-4mM\Omega+i), \quad \alpha = 4imM\Omega, \quad \beta = 1+4imM\Omega \,, \quad \gamma = 1, \tag{20}$$

$$\sigma = \ell^2+\ell-16m^2M^2\Omega^2, \quad \lambda = -4mM\Omega(4mM\Omega+i), \quad \alpha = -4imM\Omega, \quad \beta = 1-4imM\Omega, \quad \gamma = 1, \tag{21}$$

$$\sigma = \ell^2+\ell+4imM\Omega, \quad \lambda = 4imM\Omega, \quad \alpha = 4imM\Omega, \quad \beta = 1-4imM\Omega, \quad \gamma = 1 \,, \tag{22}$$

and as we can see here we have four different cases for the coefficients. Finally, radial function can be obtained as

$$\begin{aligned} R_{\ell m}(x) &= e^{\alpha x/2}(x-1)^{(\gamma+1)/2}x^{(\beta+1)/2} \\ &\times\Big[C_{\ell m1}\text{HeunC}[\sigma,\lambda,\beta,\gamma,\alpha,x] + C_{\ell m2}x^{1-\beta}\text{HeunC}[(1-\beta)(\alpha-\gamma)+\sigma,\alpha(1-\beta)+\lambda,2-\beta,\gamma,\alpha,x]\Big] \,. \end{aligned} \tag{23}$$

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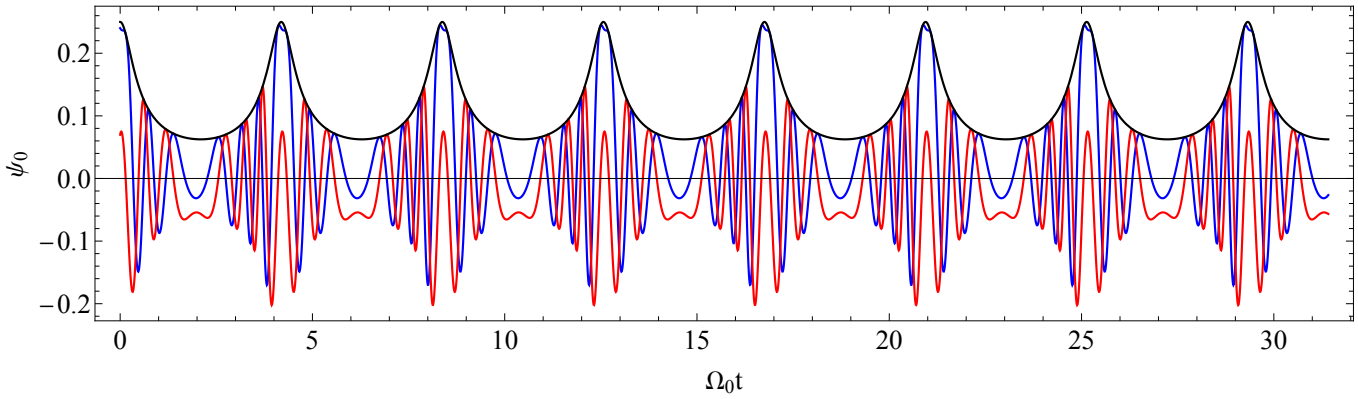


FIG. 1. Time dependence of the scalar field ψ_0 in the flat spacetime. Blue and red curves represent real and imaginary parts of the scalar field, while black line is corresponds to absolute value of the scalar field.

The total power output in mode of (ℓ, m) can be determined as

$$P_{\ell m} = \frac{1}{8\pi}|C_{\ell m}|^2|R_{\ell m}(r_0)|^2 \,. \tag{24}$$

For qualitative analyses it is also interesting to consider this problem in a flat space (i.e., $f = 1$). Hereafter introducing new wave equation $\Phi(\mathbf{r}, t) = m_0 g_s \psi_0(\mathbf{r}) e^{-i\Omega t}$, equation (9) reduces to

$$\left(\partial_t^2 - \nabla^2\right)\psi_0 = \frac{4\pi}{r^2}\delta(r - r_0)\delta(\cos\theta)\delta(\phi - \Omega t) \,, \tag{25}$$

where ∇ is the Laplacian in the spherical coordinates. Notice that the expression for Keplerian frequency $\Omega = \sqrt{M/r_0^3}$ is the same in the framework of Newtonian as well as in general relativity. It is well-known that the Green's function for the Helmholtz operator is given as [15]

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{4\pi}\frac{e^{-i\Omega|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r} - \mathbf{r}'|} \,, \tag{26}$$

and the solution for a scalar field yields

$$\psi_0(\mathbf{r}, t) = -\frac{e^{i\Omega(t-\sqrt{r^2+r_0^2+2rr_0\sin\theta\cos\Omega t})}}{\sqrt{r^2 + r_0^2 + 2rr_0\sin\theta\cos\Omega t}} \,. \tag{27}$$

In Fig.1 we showed time dependence of the scalar field in the Minkowski spacetime. Black line represents absolute value of the scalar field.

III. CONCLUSION

In this paper we have studied scalar perturbation due to geodesic motion of massive particle orbiting around the black hole described by the Schwarzschild spacetime. For simplicity, we assumed that the scalar field does not contributed in the geodesic motion. We briefly mentioned about particle motion in the Schwarzschild spacetime. We derived equation for the scalar field using Klein-Gordon equation including the source term. We have obtain fully analytical solution to Klein-Gordon equation in the Schwarzschild spacetime. We have shown that solution to angular equation can expressed in terms of the spherical harmonics while for the radial equation ti can be expressed in terms of the confluent Heun function.

Appendix A: Solution in weak field

Here we focus on the wave-like equation in the weak field approach. So that the scalar field can be expanded as

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \frac{2M}{r}\psi_1(\mathbf{r}) + \mathcal{O}(M^2/r^2) \,, \tag{A1}$$

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where Φ_0 is the solution in a flat spacetime, Φ_1 is the solution in the first leading order. Inserting expression (A1) into the equation (9), one can obtain the following wave-like equation:

$$(\nabla^2+\omega^2)\frac{\psi_1}{r} = -\frac{\pi e^{i\omega t}}{r^3}\delta(r-r_0)\delta(\cos\theta)\delta(\phi-\Omega t) - \frac{1}{r}\left[\frac{1}{r}\partial_r\left(r\partial_r\right)+\partial_t^2\right]\psi_0\,, \tag{A2}$$

and solution is

$$\psi_1 = \frac{r}{4r_0}\frac{e^{i\omega[t-\sqrt{r^2+r_0^2-2rr_0\sin\theta\cos(\phi-\Omega_0 t)}]}}{\sqrt{r^2+r_0^2-2rr_0\sin\theta\cos(\phi-\Omega_0 t)}} - \int\frac{\delta(t-|\mathbf{r}-\mathbf{r}'|)r'dr'd\cos\theta'd\phi'}{|\mathbf{r}-\mathbf{r}'|}\left[\frac{1}{r}\partial_r\left(r\partial_r\right)+\partial_t^2\right]\psi_0\,. \tag{A3}$$

Appendix B: The coordinate transformations

The Cartesian coordinates (x,y,z) are related to spherical coordinates (r,θ,ϕ) in the standard way:

$$x = r\cos\phi\sin\theta\,, \qquad y = r\sin\phi\sin\theta\,, \qquad z = r\cos\theta\,, \tag{B1}$$

$$r = \sqrt{x^2+y^2+z^2}\,, \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\,, \qquad \phi = \tan^{-1}\frac{y}{x}\,, \tag{B2}$$

where $0\le r\le\infty$, $0\le\theta\le\pi$ and $0\le\phi\le 2\pi$.

The transformation of the volume element from Cartesian to spherical coordinates is given by the following expression: $d\mathbf{r} = dxdydz = J\,drd\theta d\phi$, where J is the Jacobian determined as

$$J = \begin{bmatrix}\partial_r x & \partial_r y & \partial_r z\\ \partial_\theta x & \partial_\theta y & \partial_\theta z\\ \partial_\phi x & \partial_\phi y & \partial_\phi z\end{bmatrix} = \begin{bmatrix}\cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta\\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta\\ -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0\end{bmatrix} = r^2\sin\theta\,. \tag{B3}$$

The transformation of the Green's function from Cartesian to spherical coordinates can be done by using the following calculations:

$$\begin{aligned}|\mathbf{r}-\mathbf{r}'| &= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}\\ &= \sqrt{r^2+r'^2-2rr'\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')\right]},\end{aligned} \tag{B4}$$

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