Journal of Fundamental and Applied Research **Vol. 3**, Issue 2 (2023) **20230009**

Received: 14 April 2023 / Accepted: 26 May 2023
Published online: 1 June 2023
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Deflection of massless and massive particles by a Reissner-Nordström black hole

Yovqochev Pahlavon[1,*] and Ortiqboyev Dilmurod[2]

[1]*National University of Uzbekistan, University Street 4, Tashkent , 100174 , Uzbekistan*
[2]*Gulistan state University, Gulistan 120100, Uzbekistan*

In this paper, we study the weak gravitational lensing in the spacetime of a charged black hole geometry considering the influence of the axion-plasmon. We calculate deflection of massless particle which is light and relativistic massive particles moving in a general spherically symmetric metric in the presence of medium by using Gauss-Bonnet theorem. We discuss in detial the properties of these results and compare with solutions of other methods to verify our results. We calculate deflection angle of light in the different medium such as medium with uniform distribution of plasma and plasma containing an exponentially decaying term. Moreover, we plot dependence of the deflection angle on different parameters including black hole's charge Q, plasma frequency ω_p, axion frequency ω_ϕ, homogeneous magnetic field B_0. Finally, we compare our results which are found by using Gauss-Bonnet theorem with the expression formed by other methods.

Keywords: Gauss-Bonnet theorem, Gravitational lensing, deflection angle, Reissner-Nordström black hole, axion-plasmon, medium.

I. INTRODUCTION

Einstein's theory of general relativity reveal a lot of interesting predictions. On of them is black holes. They are not only central character of most of the high energy activities in the astronomical domain such as active-galactic nuclei, jet emissions, accretion disks, quasi-periodic oscillations and gravitational wave production to name a few, but also have fundamental role to explain the nature of space and time [1, 2].

We study the effects of plasma and magnetic field on the deflection angle by an electrically charged BH using the Reissner-Nordström (RN) BH. In particular, BHs can acquire electric charge through a combination of BH spin and the magnetic field. In literature, numerous attempts have been made to constrain the electric charge parameter using the EHT data for Sgr A* [3] and M87* [4–7], where it is concluded that the upper limit on the charge of Sgr A* has the following observational upper bound 3×10^8C. Similarly, the BH charge has been constrained via measurements of gravitational waves data obtained form the BH-BH or binary merger events [8]. All these studies confirm that the electric charge on the BH cannot be ruled out. However, the same studies reveal that Kerr and Kerr-Newman black holes are not easily distinguished in a substantial region of the EHT-constrained parameter space. This model help us investigate the effects of both plasma and magnetic field on the RN BH shadow.

The plan of the paper is as follows. In Sec. II, deflection of massive particles [9] and their correspondence with the light rays are considered with more details in presence of magnetized plasma. We discuss our results in Sec. III. Throughout the paper, we use a system of geometric units in which $G = 1 = c$. Greek indices run from 0 to 3.

II. DEFLECTION OF LIGHT AND RELATIVISTIC MASSIVE PARTICLES USING THE GAUSS-BONNET THEOREM

In this section, we shall consider the problem of computing the deflection angle for relativistic massive particles. For the same reason, let us consider the physical spacetime metric to be described by a BH surrounded by plasma medium. One can get the optical metric for investigating the deflection of light using $ds^2 = 0$, in the equatorial plane, this yields

$$dt^2 = \frac{dr^2}{f^2(r)} + \frac{r^2 d\phi^2}{f(r)}. \tag{1}$$

* E-mail: yovqochevp@gmail.com

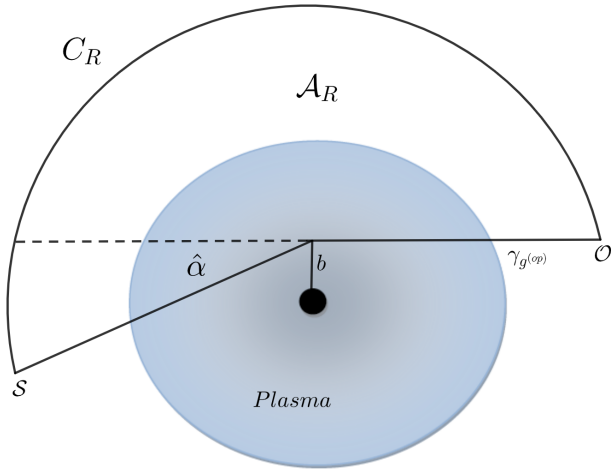


FIG. 1. Schematic representation of the optical geometry of the BH surrounded by plasma. At the points $\mathcal{O}$ and $\mathcal{S}$, the interior angles satisfy the condition $\theta_{\mathcal{O}} + \theta_{\mathcal{S}} \to \pi$.

We can use the following Gauss-Bonnet theorem (GBT) to study the deflection of light and that of relativistic massive particles.

Theorem: *Let $\mathcal{A}_R$ be a non-singular domain with boundaries $\partial\mathcal{A}_R = \gamma_{g^{(op)}} \cup C_R$ of an oriented two-dimensional optical surface S (see Fig 1) with the optical metric $g^{(op)}$. Let K and κ be the Gaussian optical curvature and the geodesic curvature, respectively. Then, the GBT in terms of the above construction is written as follows* [10]

$$\int_{\mathcal{A}_R} K\, dS + \oint_{\partial\mathcal{A}_R} \kappa\, dt + \sum_k \delta_k = 2\pi\chi(\mathcal{A}_R). \tag{2}$$

In the GBT we have the optical surface element noted as dS and the exterior angle at the corresponding k^{th} vertex noted by δ_k. It is rather interesting to see that the domain of integration is outside the light ray in the (r, ϕ) optical plane having the Euler characteristic number one, i.e., $\chi(\mathcal{A}_R) = 1$. Moreover if we introduce a smooth curve via $\gamma := \{t\} \to \mathcal{A}_R$, we can compute the geodesic optical curvature using the definition [10]

$$\kappa = g^{(op)}\left(\nabla_{\dot{\gamma}}\dot{\gamma}, \ddot{\gamma}\right). \tag{3}$$

To simplify our calculations we are going to assume the unit speed condition given by $g^{(op)}(\dot{\gamma}, \dot{\gamma}) = 1$, with $\ddot{\gamma}$ which stands for the unit acceleration vector. In the physical geometry the observer is located far away from the BH, hence by the same analogy, we can consider very large radial distance $r \equiv R \to \infty$, in such a limit, therefore we can express them in terms of the interior angles using $\theta_{\mathcal{O}} = \pi - \delta_{\mathcal{O}}$ and $\theta_{\mathcal{S}} = \pi - \delta_{\mathcal{S}}$. One can see that by construction, the two jump angles become $\pi/2$ (the jump angle at the source $\mathcal{S}$ and observer $\mathcal{O}$, respectively) and should satisfy the condition $\theta_{\mathcal{O}} + \theta_{\mathcal{S}} \to \pi$ [10]. As we know, the geodesic optical curvature for the light ray vanishes, that is $\kappa(\gamma_{g^{(op)}}) = 0$. From the GBT it follows that [10]

$$\lim_{R\to\infty} \int_0^{\pi+\hat{\alpha}} \left[\kappa \frac{dt}{d\phi}\right]_{C_R} d\phi = \pi - \lim_{R\to\infty} \int_{\mathcal{A}_R} K\, dS. \tag{4}$$

The nonzero contribution of the geodesic curvature for the curve C_R is found by using [10]

$$\kappa(C_R) = |\nabla_{\dot{C}_R}\dot{C}_R|, \tag{5}$$

and one can show the condition

$$\lim_{R\to\infty} \left(\kappa(C_R)\frac{dt}{d\phi}\right) = 1. \tag{6}$$

Note that, the above condition is true only for asymptotically flat spacetimes. For static spacetimes in the presence of an optical medium, it has been shown that the optical metric and the spatial part of the spacetime metric are related by [11]

$$g^{op}_{ij} = -\frac{n^2}{f(r)} g_{ij}, \tag{7}$$

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where $i, j = 1, 2$. In other words, the spatial projections of the light rays on the slices with $t = constant$ that solve Hamilton's equations are also spacelike geodesics of the optical metric. Let us also note that to compute the Gaussion optical curvature K, we can use the relation $K = R/2$, where R is the Ricci scalar for the optical metric.

A. Plasma medium with $\omega_{\rm p}^2 = $ const.

The simplest model corresponds to a medium with a uniform distribution of plasma. That is, the refractive index is given by

$$n^2(r) \simeq 1 - \frac{\omega_{\rm p}^2}{\omega_0^2} f(r) \left(1 + \frac{\tilde{B}_0^2}{1 - \tilde{\omega}_\varphi^2}\right). \tag{8}$$

Usings Eqs. (1), (7) and (8), we can recast the optical metric of the BH metric surrounded by the plasma as

$$\begin{aligned} dt^2 \;=\; & \left[1 - \frac{\omega_{\rm p}^2}{\omega_0^2} f(r) \left(1 + \frac{\tilde{B}_0^2}{1 - \tilde{\omega}_\varphi^2}\right)\right] \\ & \times \Big[\frac{dr^2}{f(r)^2} + \frac{r^2}{f(r)} d\phi^2\Big]. \end{aligned} \tag{9}$$

From this, we can compute the Gaussian optical curvature, and after considering series expansion around M/b and Q^2/b^2, we obtain in leading order terms

$$\begin{aligned} K \simeq & -\frac{M\left[2 - \frac{\omega_{\rm p}^2}{\omega_0^2}\left(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\right)\right]}{r^3\left[1 - \frac{\omega_{\rm p}^2}{\omega_0^2}\left(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\right)\right]^2} \\ & + \frac{Q^2\left[3 - \frac{\omega_{\rm p}^2}{\omega_0^2}\left(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\right)\right]}{r^4\left[1 - \frac{\omega_{\rm p}^2}{\omega_0^2}\left(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\right)\right]^2}. \end{aligned} \tag{10}$$

From the GBT, for the deflection angle we have

$$\hat{\alpha} = -\int_0^{\pi} \int_{\frac{b}{\sin\varphi}}^{\infty} K dS, \tag{11}$$

where we also need the expression for the surface element approximated as

$$dS \simeq r\left[1 - \frac{\omega_{\rm p}^2}{\omega_0^2}\left(1 + \frac{\tilde{B}_0^2}{1 - \tilde{\omega}_\varphi^2}\right)\right] dr d\phi. \tag{12}$$

Solving the last integral is not difficult, hence we obtain

$$\begin{aligned} \hat{\alpha} \simeq & \frac{2M}{b}\left[1 + \frac{1}{1 - \frac{\omega_{\rm p}^2}{\omega_0^2}\left(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\right)}\right] \\ & - \frac{\pi Q^2}{4b^2}\left[1 + \frac{2}{1 - \frac{\omega_{\rm p}^2}{\omega_0^2}\left(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\right)}\right] \end{aligned} \tag{13}$$

This result generalizes the deflection angle obtained in Ref. [11]. In addition to this, the dependence of the deflection angle on the axion-plasmon parameters is given in Fig.2

B. Plasma medium with $\omega_{\rm p}^2(r) = z_0/r^q$

For this particular model, the refractive index is given by

$$n^2(r) \simeq 1 - \frac{z_0}{r^q \omega_0^2} f(r) \left(1 + \frac{\tilde{B}_0^2}{1 - \tilde{\omega}_\varphi^2}\right). \tag{14}$$

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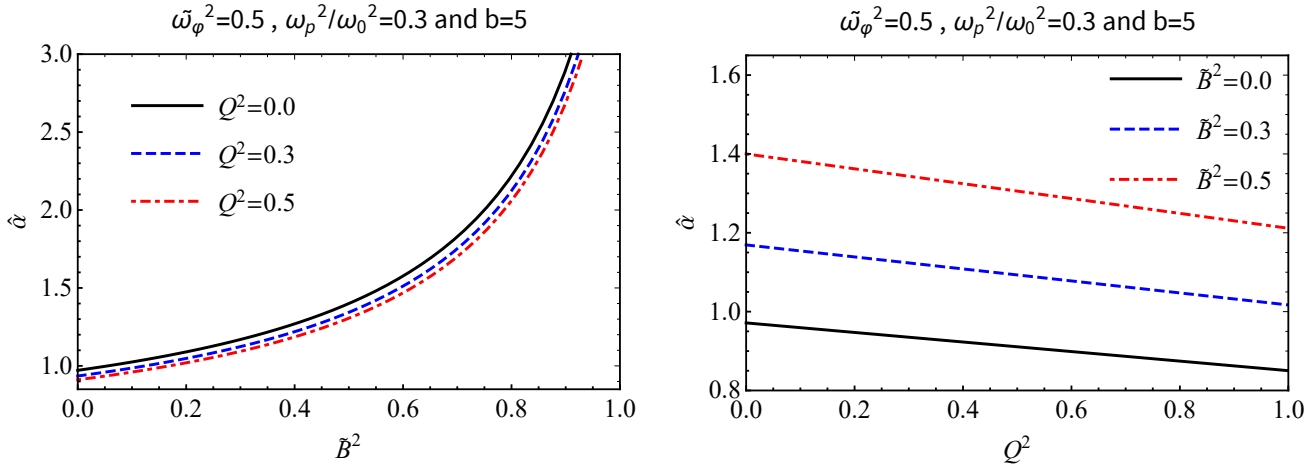


FIG. 2. Deflection angle versus the axion fluid parameters.

In our particular case, we can recast the BH metric surrounded by plasma as

$$dt^2 = \Big[1 - \frac{z_0}{r^q \omega_0^2} f(r)\Big(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\Big)\Big]\Big[\frac{dr^2}{f(r)^2} + \frac{r^2}{f(r)} d\phi^2\Big]. \tag{15}$$

Performing series expansions, for the Gaussian optical curvature we obtain in leading terms

$$K \simeq -\frac{2M}{r^3} + \frac{4Q^2}{r^4} + \frac{q^2 z_0}{2r^{q+2}\omega_0^2}\Big(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\Big). \tag{16}$$

Hence, the deflection angle is found to be

$$\hat{\alpha} = -\int_0^{\pi}\int_{\frac{b}{\sin\varphi}}^{\infty}\Big[-\frac{2M}{r^3} + \frac{4Q^2}{r^4} + \frac{q^2 z_0}{2r^{q+2}\omega_0^2}\Big(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\Big)\Big]\mathrm{d}S. \tag{17}$$

Finally, solving this integral we find

$$\hat{\alpha} \simeq \frac{4M}{b} - \frac{\pi Q^2}{b^2} - \frac{z_0\sqrt{\pi}\,\Gamma(\frac{q+1}{2})}{\omega_0^2\, b^q\, \Gamma(\frac{q}{2})}\Big(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\Big). \tag{18}$$

Again, the last expression generalizes the result found in Ref. [11].

C. Plasma medium with $\omega_{\mathbf{p}}^2(r) = b_0\, e^{-r/r_0}$

In another example we consider a plasma medium with a refractive index containing an exponentially decaying term, given by [12]

$$n^2(r) \simeq 1 - \frac{b_0 e^{-r/r_0}}{\omega_0^2} f(r)\Big(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\Big). \tag{19}$$

Here for the optical metric we obtain

$$dt^2 = \Big[1 - \frac{b_0 e^{-r/r_0}}{\omega_0^2} f(r)\Big(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\Big)\Big]\Big[\frac{dr^2}{f(r)^2} + \frac{r^2}{f(r)} d\phi^2\Big]. \tag{20}$$

For the Gaussian optical curvature we find

$$K \simeq -\frac{2M}{r^3} + \frac{4Q^2}{r^4} + \frac{b_0\, e^{-r/r_0}(r - r_0)}{2\, r\, r_0^2\, \omega_0^2}\Big(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\Big). \tag{21}$$

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The deflection angle is found

$$\hat{\alpha} \simeq \frac{4M}{b} - \frac{\pi Q^2}{b^2} - \frac{b\, b_0 K_0(b/r_0)}{r_0 \omega_0^2}\left(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\right), \tag{22}$$

where K_0 is the zeroth order modified Bessel function of the second kind. One can see that by setting $\tilde{B}_0^2 = 0$, we recover the result derived in [11].

D. Deflection of relativistic massive particles

We shall focus on the deflection of relativistic massive particles in the presence of axion-plasmon medium. To find the deflection angle for massive particles we proceed as follows. First, by following the Refs. [11, 13, 14], one can incorporate the refractive index of the medium in the optical metric, hence we can write

$$dt^2 \to d\sigma^2 = n^2(r)\left(\frac{dr^2}{f^2(r)} + \frac{r^2 d\phi^2}{f(r)}\right), \tag{23}$$

in the last equation we have used Eq. (7) to introduce the refractive index in the metric. Secondly, we use the correspondence between the motion of photons in plasma and massive particles i.e. we can identify the rest mass of the particle with the frequency of the plasma, and the energy of the particle with the photon frequency, hence (with $\hbar = 1$)

$$\omega_{\rm p} \longrightarrow m_0, \;\; \omega_0 \longrightarrow E. \tag{24}$$

For the refractive index we can write

$$n^2(r) \simeq 1 - \frac{m_0^2}{E_\infty^2} f(r)\left(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\right). \tag{25}$$

The relativistic particle with velocity v has energy

$$E_\infty = \frac{m_0}{\sqrt{1-v^2}}, \tag{26}$$

as measured far away from the observer at spatial infinity. In addition, let us assume that the particle has an angular momentum given by

$$J = \frac{m_0 v\, b}{\sqrt{1-v^2}}, \tag{27}$$

here b is an impact parameter of the massive particle. Combining Eqs. (25), (26) and (27), we find

$$n^2(r) \simeq 1 - (1-v^2) f(r)\left(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\phi^2}\right) \tag{28}$$

That is, we can recast the purely optical metric in presence of axion-plasmon medium in the equatorial plane as

$$d\sigma^2 = \left[1 - (1-v^2) f(r)\left(1 + \frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2}\right)\right]\left[\frac{dr^2}{f^2(r)} + \frac{r^2 d\phi^2}{f(r)}\right]. \tag{29}$$

The Gaussian optical curvature from the metric (29) is computed after we perform series expansions around M/b and Q^2/b^2, we get

$$\begin{aligned} K \simeq & -\frac{M\left[1+v^2-(\frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2})(1-v^2)\right]}{r^3\left[v^2-(\frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2})(1-v^2)\right]^2} \\ & + \frac{Q^2\left[2+v^2-(\frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2})(1-v^2)\right]}{r^4\left[v^2-(\frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2})(1-v^2)\right]^2}, \end{aligned} \tag{30}$$

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in leading order terms. Notice an interesting consequence in the last equation in the limit $v \to 0$ and $\tilde{B}_0 \to 0$, yielding an apparent singularity in K. This apparent singularity shows that, we need to restrict our analyses only for relativistic particles, namely, the particles speed belongs to the following interval $0 < v \leq 1$ (with $c = 1$), while this also suggests that for nonrelativistic motions one must develop a different or more general setup. For the geodesic deviation having large radial coordinate R, yields

$$\lim_{R\to\infty} \kappa(C_R) \to \frac{1}{\left(v^2 - (\frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2})(1-v^2)\right) R}. \tag{31}$$

Using the metric (29), for $r = R$ held constant we find that

$$\lim_{R\to\infty} d\sigma \to \left(v^2 - (\frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2})(1-v^2)\right) R\, d\phi. \tag{32}$$

This leads to the expected condition

$$\lim_{R\to\infty} \left(\kappa(C_R)\frac{d\sigma}{d\phi}\right) = 1. \tag{33}$$

To compute the deflection angle we need to use

$$\hat{\alpha} = -\int_0^{\pi}\int_{\frac{b}{\sin\varphi}}^{\infty} K dS. \tag{34}$$

where we need to integrate over the optical domain with the approximated surface element

$$dS \simeq r\left(v^2 - (\frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2})(1-v^2)\right) dr d\phi. \tag{35}$$

Finally, evaluating this integral we find

$$\begin{aligned}\hat{\alpha} \simeq \frac{2M}{b}&\left[1 + \frac{1}{\left(v^2 - (\frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2})(1-v^2)\right)}\right] - \\ &-\frac{\pi Q^2}{4b^2}\left[1 + \frac{2}{\left(v^2 - (\frac{\tilde{B}_0^2}{1-\tilde{\omega}_\varphi^2})(1-v^2)\right)}\right].\end{aligned} \tag{36}$$

If we set $\tilde{B}_0^2 = 0$, we obtain the result earlier found in Ref. [11].

III. CONCLUSIONS

In this work, we investigated the axion-plasmon effect on the optical properties of the RN BH. Specifically, we studied the gravitational lensing and calculated the deflection angle by a charged black using Gauss-Bonet theorem. Moreover, we generalize the expression for deflection angle[15] by adding axion-plasmon term. We found that our results coincide with the expression for the deflection angle obtained by the standard geodesic methods. Finally, we plotted the dependence of the deflection angle on axion-plasmon parameters and alson black hole. It can be seen from Fig.2 that with increasing black hole's charge the deflection angle decreases. But, We find that for an observer far away from the Reissner–Nordström black hole with increasing axion-plasmon, the deflection angle increases.

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