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Nodal precession frequency of free charged test particle orbiting around compact object influence of magnetic field

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We consider the fundamental frequencies, namely, Keplerian and nodal (radial and vertical) frequencies, in the Schwarzschild spacetime in the presence of external magnetic field to study the so-called kilohertz quasi-periodic oscillations from low-mass X-ray binary systems. It is shown that, in the presence of magnetic field, the observational data for kilohertz quasi-periodic oscillations in terms of the fundamental frequencies of test particles in the Schwarzschild spacetime. By constructing the relation between the upper and lower frequencies and exploiting the quasi-periodic oscillation data of the some sources we perform the linear model fit procedure and estimate the mass of the central object.

I. INTRODUCTION

The nodal frequency of a particle orbiting a black hole refers to the frequency at which the line of nodes (the intersection of the particle's orbital plane with a reference plane) precesses around the black hole. This precession is caused by the curvature of spacetime near the black hole. The nodal frequency depends on several factors, including the mass and spin of the black hole, the eccentricity and inclination of the particle's orbit, and the distance of the particle from the black hole. For a circular orbit around a non-spinning black hole, the nodal frequency is zero, as the line of nodes does not precess. However, for an eccentric orbit or an orbit around a spinning black hole, the nodal frequency will be non-zero. The nodal frequency can be calculated using Kepler's laws of planetary motion and the equations of motion for a test particle in the gravitational field of a black hole. However, the calculations can be complex and require advanced mathematical techniques, such as perturbation theory and numerical simulations.

Quasi-periodic oscillations (QPOs) are observed in the X-ray emission from compact objects such as black holes and neutron stars. These oscillations are thought to arise from variations in the accretion flow around the compact object, and their study can provide insights into the properties of the accretion flow and the nature of the compact object itself. QPOs are observed over a range of frequencies, typically from a few Hz up to several hundred Hz. The frequencies of the QPOs depend on the mass and spin of the compact object, the properties of the accretion flow, and the distance of the emitting region from the compact object. In the case of neutron stars, the QPO frequencies can be related to the properties of the neutron star itself, such as its mass and radius, as well as the properties of the accretion flow onto the neutron star. The study of QPOs is an active area of research in astrophysics, and ongoing observations and theoretical modeling are helping to improve our understanding of the physical processes that give rise to these oscillations.

According to the models studying and explaining the physical properties and origin of the quasi-periodic oscillations (QPOs) discovered in low-mass X-ray binary (LMXRB) systems, the QPOs data could provide independent information about the mass of the central compact object (a white dwarf, neutron star and black hole), hence give some clues about the structure of accretion disks [1, 2]. Moreover it is believed that the QPO data might allow one to test the effects of General Relativity (GR) in the strong field regime [3]. There are several models attempting to explain the QPOs in the literature [4–12]. In all of these models the key point is the assumption that the QPOs originate in the orbits around the compact object onto which the matter accretes. In this note I will refer to the model that assumes that the QPOs are caused by the fundamental (epicyclic) frequencies associated with the orbital motion of test particle in the presence of magnetic field [13–16].

Before start our calculation let us introduce main parameters of neutron star which we use further: $\boldsymbol{M}$ – *mass of neutron star*, $\boldsymbol{R}$ – *radius of neutron star*, $\boldsymbol{\Omega}$ – *angular velocity*, $\boldsymbol{I}$ – *moment of inertia*, $\boldsymbol{J}$ – *angular momentum*, $\boldsymbol{\sigma}$ – *electric conductivity*, $\boldsymbol{\rho_e}$

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– *charge density*, $\boldsymbol{B_0}$ – *surface magnetic field*, $\boldsymbol{L}$ – *luminosity* and with the definitions of the *stellar compactness*

$$\epsilon = \frac{2M}{R} \,,$$

and *the magnetic dipole moment*

$$\mu = \frac{1}{2} B_0 R^3 \,,$$

In the Boyer-Lindquist coordinate (t, r, θ, ϕ) the spacetime outside of the slowly rotating magnetized neutron star can be expressed by following line element [17]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) - 2\omega(r) r^2 \sin^2\theta dt d\phi \,, \tag{1}$$

with the lapse function

$$f(r) = 1 - \frac{2M}{r}, \qquad r \geq R \,,$$

where M and R are, respectively, the total mass and radius of the neutron star and $\omega(r) = 2aM/r^3$ is an angular velocity in the dragging frame or the Lense-Terring angular velocity with the specific angular momentum $a = J/M$, where J is the total angular momentum of the rotating compact object.

Before start our calculation let us first consider inner most stable circular orbit (ISCO) radius for test particle around static $(a = 0)$ neutron star which is exactly same as in Schwarzschild space:

$$r_{\rm ISCO} = 6M \,, \quad \text{or} \quad r_{\rm ISCO} \simeq 9\left(\frac{M}{M_\odot}\right) \text{km} \,, \tag{2}$$

Note that in case of neutron star, ISCO radii for test particle orbiting around neutron star will be order of stellar radius $r_{\rm ISCO} \sim R$, from equation (2) we can immediately find more precise expression:

$$r_{\rm ISCO} = 3\epsilon R \,, \tag{3}$$

which means that in case of $\epsilon \leq 1/3$ ISCO radii don't exist, because in this case ISCO radius is smaller than stellar radius. From here we can conclude that ISCO radii exist only when stellar compactness is in the range of $0.33 \leq \epsilon \leq 0.5$. Figure 1 draws dependence of ISCO radii from the compactness of star. We can see that ISCO radii can reach up to one and half stellar radius $\sim 1.5R$ for neutron star with compactness $\epsilon \sim 0.5$.

Since we consider slowly rotating limit of Kerr spacetime (1) for neutron star then we should also take into account rotation effect in ISCO radii for test particle. To do this we simply take expression of ISCO radii in Kerr spacetime

$$r_{\rm ISCO} = \left[3 + Z_2 \pm \sqrt{(3+Z_1)(3-Z_1+Z_2)}\right] M \,, \tag{4}$$

where $Z_1 = 1 - \sqrt[3]{1-a_*^2}\left(\sqrt[3]{1+a_*} + \sqrt[3]{1-a_*}\right)$ and $Z_2 = \sqrt{3a_*^2 + Z_1}$, with $a_* = a/M$ as the rotation parameter. In the case of slowly rotating limit expression takes a form:

$$r_{\rm ISCO} = \left[6 \pm 4\sqrt{\frac{2}{3}}\frac{a}{M}\right] M \simeq \left[3 \pm 2\sqrt{\frac{2}{3}}\frac{a}{M}\right] \epsilon R \,. \tag{5}$$

II. CHARGED PARTICLE MOTION AROUND MAGNETIZED COMPACT OBJECT

The vector potential of electromagnetic field is defined as

$$A_\phi(r,\theta) = \begin{cases} \frac{1}{2} B r^2 \sin^2\theta \,, & \text{Uniform magnetic field} \,, \\ -\frac{3\mu}{8M^3}\mathcal{F}(r) r^2 \sin^2\theta \,, & \text{Dipole magnetic field} \,, \end{cases} \tag{6}$$

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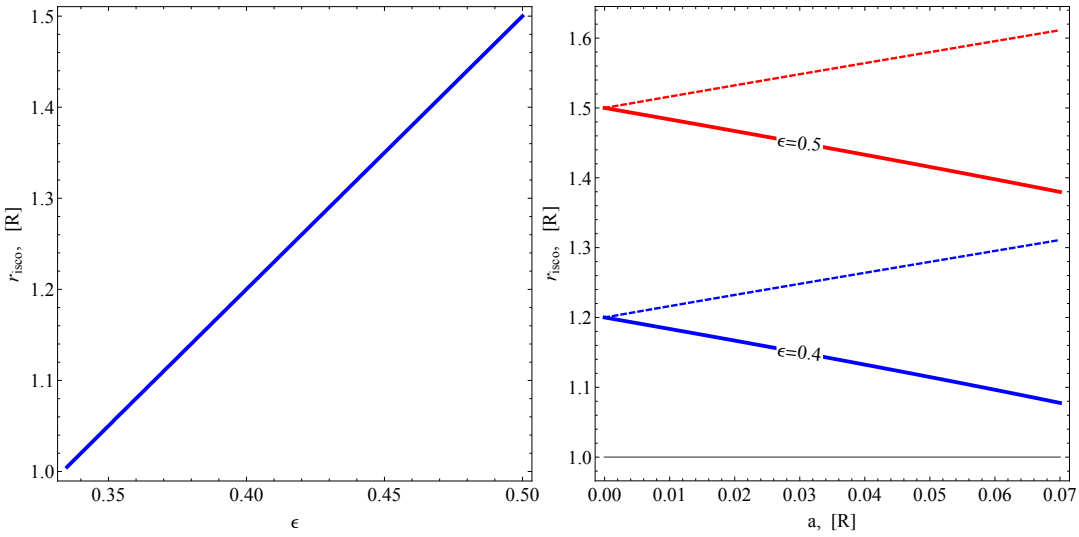


FIG. 1. (Left panel) Dependence of ISCO radii for neutral particle from compactness of NS, (Right panel) dependence of ISCO radii for neutral particle from spin parameter for different values of stellar compactness.

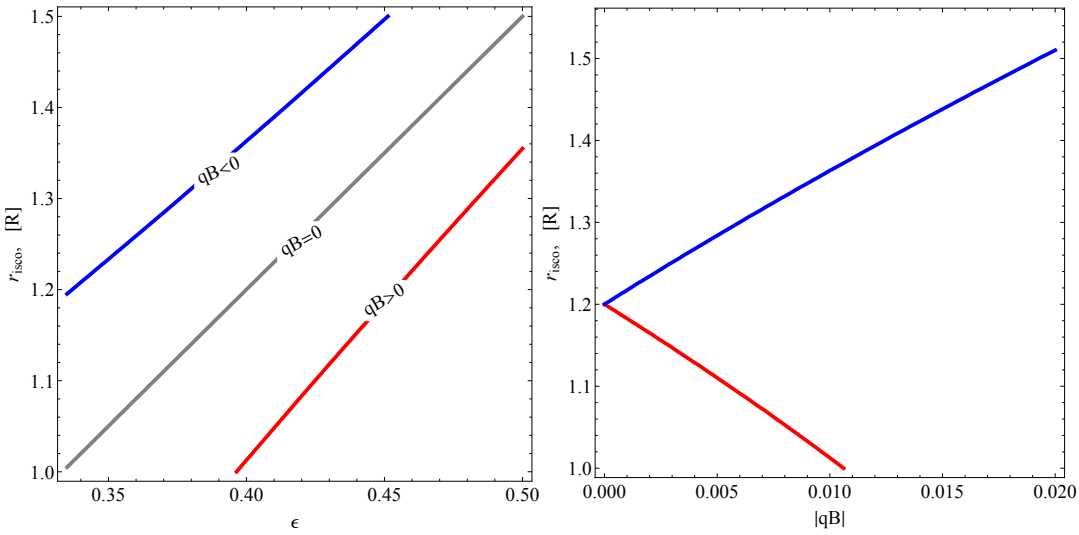


FIG. 2. (Lest panel) Dependence of ISCO radii for charged particle from stellar compactness for different values of interaction parameter qB. (Right panel) Dependence of ISCO radii for charged particle from interaction parameter qB at $\epsilon = 0.4$.

where B is the uniform magnetic field, μ is the magnetic dipole moment and the function $\mathcal{F}(r)$ is defined as

$$\mathcal{F}(r) = \ln\left(1 - \frac{2M}{r}\right) + \frac{2M}{r}\left(1 + \frac{M}{r}\right) \simeq -\frac{8M^3}{r^3} + \mathcal{O}\left(\frac{M^4}{r^4}\right) . \tag{7}$$

Since the spacetime metric (1) independent from coordinates t and ϕ there exist conserved quantities corresponding to these coordinates which are the energy E and the angular momentum L of charged particle:

$$P_t = mu_t = -E = \text{const} , \qquad u^t = \frac{\mathcal{E}}{f(r)} , \tag{8}$$

$$P_\phi = mu_\phi + qA_\phi = L = \text{const} , \quad u^\phi = \frac{1}{r^2 \sin^2\theta}\left(\mathcal{L} - \frac{q}{m}A_\phi\right) , \tag{9}$$

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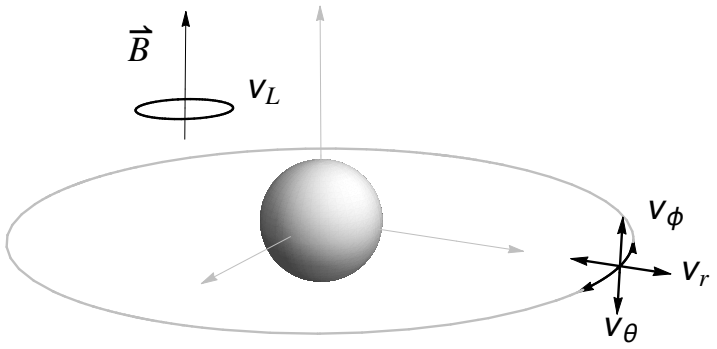


FIG. 3. Schematic view of oscillating of particle around compact object in the presence of magnetic field.

where $\mathcal{E}$ and $\mathcal{L}$ are, respectively, specific energy and specific angular momentum of test particle. Particle motion can be described by the following equation simple equation $g_{\alpha\beta}u^\alpha u^\beta = -1$ or more precisely

$$\mathcal{E}^2 = \dot{r}^2 + f(r)r^2\dot{\theta}^2 + V_{\rm eff}(r,\theta)\ , \tag{10}$$

where $V_{\rm eff}$ is effective potential

$$V_{\rm eff}(r,\theta) = f(r)\begin{cases} 1 + \left(\frac{\mathcal{L}}{r\sin\theta} - \frac{\mathcal{B}r}{M}\sin\theta\right)^2\ , & \text{Uniform magnetic field}\ , \\ 1 + \left(\frac{\mathcal{L}}{r\sin\theta} - \frac{kr}{M}\mathcal{F}(r)\sin\theta\right)^2\ , & \text{Dipole magnetic field}\ , \end{cases} \tag{11}$$

with the following parameters

$$\mathcal{B} = \frac{qBM}{m}\ , \qquad k = -\frac{3q\mu}{8mM^2}\ .$$

From equation (11) we easily can see that the effective potential for charged particle in case of dipole magnetic field is more general in comparison with uniform magnetic field case. That is why from now we consider particle motion in dipole magnetic field. Careful analyse of the effective potential shows that $\partial_\theta V_{\rm eff}(r,\theta) = 0$ gives $\theta_0 = \pi/2$ which corresponds to an equatorial plane. Extremum point of the effective potential with respect to radial coordinate r_0 can be found from the following condition, $\partial_r V_{\rm eff}(r,\theta) = 0$,

$$\mathcal{L}_\pm(r) \ = \ r\left[\pm\sqrt{\frac{M}{r-3M} + k^2\mathcal{G}^2(r)} - k\mathcal{G}(r) + \frac{kr}{M}\mathcal{F}(r)\right]\ ,$$

substituting it into equation (11) we obtain extrema for specific energy

$$\mathcal{E}_\pm(r) \ = \ \sqrt{f(r)\left[1 + \left(\pm\sqrt{\frac{M}{r-3M} + k^2\mathcal{G}^2(r)} - k\mathcal{G}(r)\right)^2\right]}\ ,$$

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where function $\mathcal{G}(r)$ is defined as

$$\mathcal{G}(r) = \frac{r^2 f(r)\left(r\mathcal{F}'(r) + 2\mathcal{F}(r)\right)}{2M(r-3M)}\,, \tag{12}$$

In the presence of uniform magnetic field specific energy and specific angular momentum for charged particle can be obtained by putting $\mathcal{F}(r) = \mathcal{B}/k$ and $\mathcal{F}'(r) = 0$.

Now we are a position to calculate angular velocity for distant observer Ω, so-called Keplerean angular velocity Ω_K, to determinate it we use the following standard approach

$$\Omega = \frac{d\phi}{dt} = \frac{u^\phi}{u^t} = \frac{f(r)}{\mathcal{E}}\left(\frac{\mathcal{L}}{r^2} - \frac{k}{M}\mathcal{F}(r)\right)\,. \tag{13}$$

or

$$\Omega_\pm = \frac{\sqrt{f(r)}\left(\pm\sqrt{\frac{M}{r-3M} + k^2\mathcal{G}^2(r)} - k\mathcal{G}(r)\right)}{r\sqrt{1 + \left(\pm\sqrt{\frac{M}{r-3M} + k^2\mathcal{G}^2(r)} - k\mathcal{G}(r)\right)^2}}\,, \tag{14}$$

Here question arises, which sign is correct? For small value of k parameter equation (14) takes a form

$$\Omega_\pm = \pm\sqrt{\frac{M}{r^3}} - \frac{k\left(r\mathcal{F}'(r) + 2\mathcal{F}(r)\right)}{2M}\sqrt{1 - \frac{3M}{r}} \pm \frac{k^2\sqrt{r}(r-5M)\left(rF'(r) + 2F(r)\right)^2}{8M^{5/2}} + \mathcal{O}(k^3)\,, \tag{15}$$

It is well know that in background in pure Schwarzschild metric the Keplerean frequency has the form $\Omega_K = \sqrt{M/r^3}$, which means one have to take plus sign $+$ in all expressions above. Finally, in the presence of external dipole magnetic field Keplerean frequency can be expressed as

$$\Omega_K(k) = \frac{\sqrt{f(r)}\left(\sqrt{\frac{M}{r-3M} + k^2\mathcal{G}^2(r)} - k\mathcal{G}(r)\right)}{r\sqrt{1 + \left(\sqrt{\frac{M}{r-3M} + k^2\mathcal{G}^2(r)} - k\mathcal{G}(r)\right)^2}}\,. \tag{16}$$

Figure 2 represents the ISCO position of charged particle around neutron star, for attractive, repulsive and null Lorentz forces.

III. CHARGED PARTICLE MOTION AROUND MAGNETIZED NEUTRON STAR

In this section we investigate charged particle motion in the field of slowly rotating magnetized neutron star. Spacetime outside of neutron star is given by equation (1). Since neutron star is magnetized it produce strong magnetic field:

The nonzero component of the vector potential of electromagnetic field are given by (See appendix A)

$$A_t = \frac{2a\mu}{r^3} - aN^2F(r) - \omega A_\phi - \frac{1}{6}rN\Big[f_2(r) + f_4(r)\Big]\left(3\cos^2\theta - 1\right)\,, \tag{17}$$

$$A_\phi = \frac{1}{2}r^2F(r)\sin^2\theta\,. \tag{18}$$

Here we will study the motion of charged particle around magnetized relativistic gravitational object using the Hamilton-Jacobi equation

$$g^{\mu\nu}\Big(\frac{\partial S}{\partial x^\mu} + qA_\mu\Big)\Big(\frac{\partial S}{\partial x^\nu} + qA_\nu\Big) = -m^2\,, \tag{19}$$

Since, t and ϕ are the Killing variables, the equation can be written in the following form for the case $\chi = 0$

$$S = -\mathcal{E}t + \mathcal{L}\phi + S_r(r) + S_\theta(\theta)\,, \tag{20}$$

One can easily find Effective potential for the particle in the equatorial plane (at $\theta = \pi/2$) by putting Eq.20 into Eq.19 (at $\dot{r} = 0$ $\dot{\theta} = 0$ $\mathcal{E} = V_{\text{eff}}$) in the following form

$$V_{\text{eff}}(r,\theta) = \mathcal{L}\omega + N\sqrt{1 + \left(\frac{\mathcal{L}}{r\sin\theta} + \frac{q}{2m}F(r)r\sin\theta\right)^2} \tag{21}$$

$$- \frac{q}{m}\left[\frac{rN}{6}\Big(f_2(r) + f_4(r)\Big)(3\cos^2\theta - 1) - \frac{2a\mu}{r^3} + aN^2F(r)\right]\,. \tag{22}$$

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IV. FUNDAMENTAL FREQUENCIES

In this section we discuss nodal frequencies of charged particle near the magnetized neutron star which related to quasi-periodic oscillation at given circular orbit. The schematic picture of such motion can be illustrated in Fig. 3. Before going further let us expand of the effective potential:

$$\begin{aligned}\delta V_{\rm eff}(r,\theta) &= V_{\rm eff}(r+\delta r,\theta+\delta\theta)-V_{\rm eff}(r,\theta)\\ &= \partial_r V_{\rm eff}(r,\theta)\delta r+\partial_\theta V_{\rm eff}(r,\theta)\delta\theta\\ &+ \frac{1}{2}\partial_r^2 V_{\rm eff}(r,\theta)\delta r^2+\partial_r\partial_\theta V_{\rm eff}(r,\theta)\delta r\delta\theta+\frac{1}{2}\partial_\theta^2 V_{\rm eff}(r,\theta)\delta\theta^2\\ &= \frac{1}{2}\partial_r^2 V_{\rm eff}(r,\theta)\delta r^2+\frac{1}{2}\partial_\theta^2 V_{\rm eff}(r,\theta)\delta\theta^2 \end{aligned} \tag{23}$$

here we have used the following relations $\partial_r V_{\rm eff}(r,\theta)=0$ and $\partial_\theta V_{\rm eff}(r,\theta)=0$. Harmonic oscillations around the point r_0,θ_0, where the effective potential $V_{\rm eff}$ is minimum with displacement $\delta r=r-r_0,\quad \delta\theta=\theta-\theta_0$ an be governed by the following equations

$$\delta\ddot{r}+\omega_{\rm r}^2\delta r=0,\qquad \delta\ddot{\theta}+\omega_\theta^2\delta r=0 \tag{24}$$

where the dots denote derivative with respect to the proper time of test particles and the frequencies ω_r and ω_θ can be found as

$$\omega_{\rm r}^2=f(r)\frac{\partial^2 V_{\rm eff}}{\partial r^2}\,,\qquad \omega_\theta^2=\frac{1}{r^2}\frac{\partial^2 V_{\rm eff}}{\partial\theta^2}\,, \tag{25}$$

In order to calculate the angular frequencies for distant observers one have take into account the gravitational redshift, which means

$$\Omega_{(r,\theta)}=\frac{1}{u^t}\omega_{(r,\theta)}=\omega_{(r,\theta)}\frac{f(r)}{\mathcal{E}}\,, \tag{26}$$

Finally, in the presence of external magnetic field the angular frequencies for distant observer can be expressed as

$$\begin{aligned}\Omega_r=\Omega_K\Big[&\frac{f''(r)}{2f(r)}+\frac{k^2\left(rF'(r)+2F(r)\right)^2}{M^2}+\frac{\left(r^2f''(r)-4rf'(r)+6f(r)\right)}{2r^4f(r)}\\ &+\frac{k\left(f(r)\left(6F(r)-r^2F''(r)\right)-2rf'(r)\left(rF'(r)+2F(r)\right)\right)}{Mr^2f(r)}\Big]\,,\end{aligned} \tag{27}$$

$$\Omega_\theta=\Omega_K\sqrt{1+\frac{2kM\mathcal{F}(r)}{r\sqrt{\sqrt{\frac{M}{r-3M}+k^2\mathcal{G}^2(r)}-k\mathcal{G}(r)}}}\,, \tag{28}$$

and for a small value of magnetic coupling parameter the angular frequencies take the form

$$\Omega_r=\sqrt{\frac{M}{r^3}\left(1-\frac{6M}{r}\right)}\left(1-\frac{kr^2(r-2M)\sqrt{r-3M}\left(r\mathcal{F}''(r)+3\mathcal{F}'(r)\right)}{2M^{3/2}(r-6M)}\right)+\mathcal{O}(k^2)\,, \tag{29}$$

$$\Omega_\theta=\sqrt{\frac{M}{r^3}}-\frac{k\sqrt{r(r-3M)}}{2M}\mathcal{F}'(r)+\mathcal{O}(k^2)\,, \tag{30}$$

In fact that the measurable quantity should be dimensional, that is why here we can write expressions for frequencies in the form:

$$\nu_i=\frac{1}{2\pi}\frac{c^3}{GM}\Omega_i\,,\qquad i=(r,\theta,\phi)\,, \tag{31}$$

In this astrophysically relevant scenario, we consider a slowly rotating neutron star that possesses a dipole magnetic field and is surrounded by a thin accretion disc consisting of charged test particles moving along nearly circular geodesics in the equatorial plane. We assume that the dipole magnetic field is fully dominant in the total electromagnetic field in the vicinity of the star, such that the influence of the magnetic field generated by the currents in the disc and the total disc charge can be neglected. This assumption holds true if the specific charge of the material in the disc is very low. Additionally, this configuration enables

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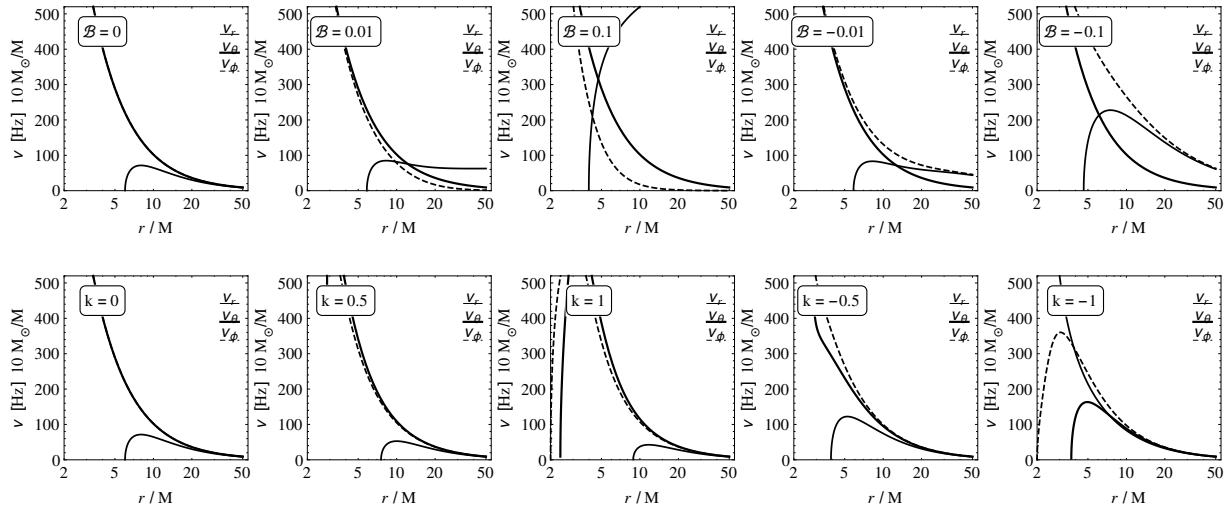


FIG. 4. (Top panel) Radial dependence of nodal frequencies for different values of interaction parameter $\mathcal{B}$. (Bottom panel) Dependence of ISCO radii for charged particle from interaction parameter k.

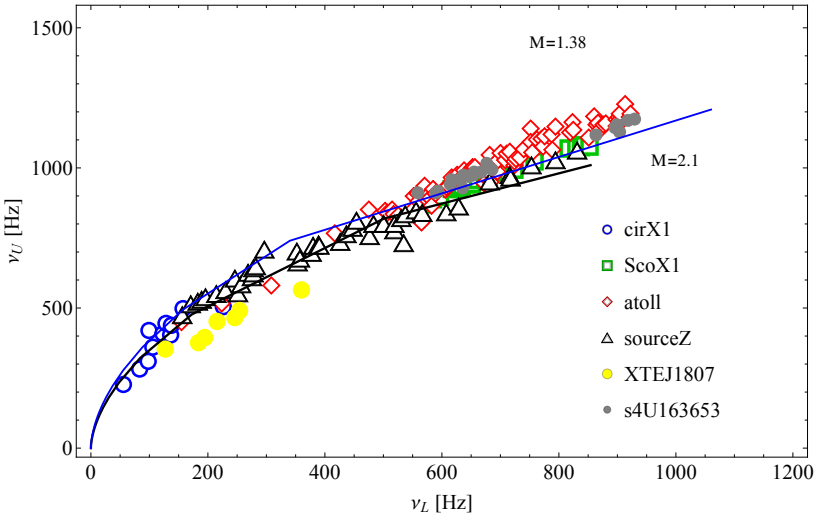


FIG. 5. Best fit of QPO frequencies in the presence of dipole magnetic field for $k = \pm 0.4$ and $M = 1.4M_\odot$.

us to use the test particle approximation, which is consistent with the assumed small non-geodesic corrections to geodesic orbital motion. Radial dependence of the fundamental frequencies in the presence of the uniform and dipole magnetic field are illustrayed in Fig. 4. Considering the RP model in line with the corrected frequencies introduced above, the new fits can provide the characteristic neutron star mass close to $M \sim 1.4M_\odot$. In figure 5 we illustrate this finding when the innermost stable circular orbit is shifted to $r = 6M$. Such a rough fit for a wide set of LMXB is shown together with the fits for the pure Schwarzschild geodesic cases with $M = 2M_\odot$ and $M = 1.4M_\odot$.

V. CONCLUSIONS

This paper aims to investigate the impact of the Lorentz force generated by a magnetic field of a neutron star on the quasi-circular, epicyclic orbital motion. Specifically, we examine the behavior of the non-geodesic orbital and epicyclic frequencies with respect to the neutron star magnetic dipole moment and the specific charge of the orbiting matter.

In general, the Lorentz force can be attractive or repulsive, depending on the sign of the orbiting particle's specific charge and the magnetic dipole moment and orbital velocity orientations. If the specific charge is large enough, both types of force can allow for circular orbits at all radii above the horizon. In the attractive region, a discontinuity appears, with only unstable circular

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orbits existing under the circular photon orbit at $r_{\rm ph} = 3M$, being oppositely oriented to those located above $r_{\rm ph}$. Surprisingly, in the repulsive region, the stable circular orbits associated with the radial and vertical epicyclic oscillations can extend below the circular photon orbit radius $r_{\rm ph}$. There exists a critical charge parameter for a given magnetic moment of neutron star. In contrast, inside the attractive region, the ISCO orbits for charged particle always appear below $r = 6M$. Therefore, one can conclude that the Lorentz force strongly affects the location of the inner edge of the thin accretion disk.

In both the repulsive and attractive regions of the magnetic interaction, the behavior of the orbital and epicyclic frequency profiles is quite complicated, giving rise to two separated regions of circular orbital motion for certain values of charge parameter. Generally, for stable circular orbits in the repulsive region, Ω_r increases with growing specific charge, while both Ω_K and Ω_θ decrease. In the attractive region, on the other hand, the frequencies exhibit the opposite behavior. For both regions of the magnetic interaction and astrophysically relevant values of the radial coordinate, the sensitivity of the radial epicyclic frequency Ω_r to charge parameter is significantly higher than the sensitivity of the two remaining frequencies. The presence of the dipole magnetic field also violates the Ω_θ equality corresponding to the spherical symmetry of the background Schwarzschild geometry. As a result, nodal precession of the orbital motion plane arises, having an opposite phase for attractive and repulsive magnetic interaction.

Appendix A: Derivation of vector potential of electromagnetic fields

The general form of the pair of the general relativistic Maxwell equations are given by

$$\nabla_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) = 4\pi J^\nu \,, \tag{A1}$$

$$\nabla^*_\mu F^{\mu\nu} = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}^* F^{\mu\nu}\right) = 0 \,, \tag{A2}$$

where $g = \det|g_{\mu\nu}|$ is the determinant of the metric tensor which has a form $g = -r^4 sin^2\theta$ in the slowly rotating limit, $F_{\mu\nu}$ is the electromagnetic field tensor which can be expressed in terms of the electric and the magnetic in the following form

$$F_{\mu\nu} \equiv 2u_{[\mu}E_{\nu]} + \eta_{\mu\nu\alpha\beta}\, u^\alpha B^\beta \,, \tag{A3}$$

with

$$\eta_{\mu\nu\alpha\beta} = \sqrt{-g}\,\varepsilon_{\mu\nu\alpha\beta} \,, \quad \eta^{\mu\nu\alpha\beta} = -\frac{1}{\sqrt{-g}}\,\varepsilon_{\mu\nu\alpha\beta} \,, \tag{A4}$$

where $\varepsilon_{\alpha\beta\gamma\delta}$ is fully Levi-Civita symbol in four dimensional space. In equation (A3) u^μ is the four-velocities of an observers that can be expressed in the ZAMO's (zero angular momentum observer's) frame of reference in the following form

$$(u^\mu)_{\rm ZAMO} = \frac{1}{N}(1,0,0,\omega), \;\; (u_\mu)_{\rm ZAMO} = -N(1,0,0,0). \tag{A5}$$

Note, that the electric and the magnetic fields are can be written in terms of the electromagnetic field tensor in the following form

$$E_\mu = F_{\mu\nu}u^\nu, \quad B^\mu = \frac{1}{2}\eta^{\mu\nu\alpha\beta}u_\nu F_{\alpha\beta}. \tag{A6}$$

It is always useful to write measurable value in physical system. Here we will introduce the projection of electrical and magnetic field $E_{\hat\alpha} = e^\beta_{\hat\alpha}E_\beta$ and $B^{\hat\alpha} = e^{\hat\alpha}_\beta B^\beta$, respectively. On the other hand mathematical simplest is $E_{\hat\alpha} = E^{\hat\alpha}$, $B_{\hat\alpha} = B^{\hat\alpha}$, and we can only measure strength of tangential projection of B^α and E^α in observational experiments.

The tedrads are

$$\begin{aligned}
\boldsymbol{e}^\alpha_{\hat t} &= \frac{1}{N}(1,0,0,\omega) \,, \qquad \boldsymbol{e}^{\hat t}_\alpha = N(1,0,0,\omega) \,, \\
\boldsymbol{e}^\alpha_{\hat r} &= N(0,1,0,0) \,, \qquad \boldsymbol{e}^{\hat r}_\alpha = \frac{1}{N}(0,1,0,0) \,, \\
\boldsymbol{e}^\alpha_{\hat\theta} &= \frac{1}{r}(0,0,1,0) \,, \qquad \boldsymbol{e}^{\hat\theta}_\alpha = r(0,0,1,0) \,, \\
\boldsymbol{e}^\alpha_{\hat\varphi} &= \frac{1}{r\sin\theta}(0,0,0,1) \,, \qquad \boldsymbol{e}^{\hat\varphi}_\alpha = r\sin\theta(-\omega,0,0,1) \,.
\end{aligned} \tag{A7}$$

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We assume that four-vector potentials does not depend on two t and φ coordinates due to the electromagnetic field components (E^{α} and B^{α}) not depending on such coordinates, so partial derivatives of four-potentials with respect to two coordinates become zero i.e $\partial A_{\alpha}/\partial x^{0} \equiv 0$ and $\partial A_{\alpha}/\partial x^{3} \equiv 0$, as well as $E_{\varphi} \equiv 0$ is explained in equation (126) in Ref. [17] at the zero inclination angle.

On the basis of expression (A3) one can have equations for non-zero components the electromagnetic field tensor as

$$F_{rt} = \frac{\partial A_0}{\partial r}, F_{\theta t} = \frac{\partial A_0}{\partial \theta}, \tag{A8}$$

and

$$F_{r\varphi} = \frac{\partial A_3}{\partial r}, F_{\theta\varphi} = \frac{\partial A_3}{\partial \theta}. \tag{A9}$$

On the other hand in case of non-zero inclination angle ($\chi \neq 0$), the components of the stationary magnetic and electric fields of the slowly rotating magnetized neutron star can be expressed in the following form [17]

$$B^{\hat{r}}(r,\theta,\phi,t) = F(r)\left(\cos\chi\cos\theta + \sin\chi\sin\theta\cos\lambda\right) , \tag{A10}$$

$$B^{\hat{\theta}}(r,\theta,\phi,t) = G(r)\left(\cos\chi\sin\theta - \sin\chi\cos\theta\cos\lambda\right) , \tag{A11}$$

$$B^{\hat{\phi}}(r,\theta,\phi,t) = G(r)\sin\chi\sin\lambda , \tag{A12}$$

and

$$E^{\hat{r}}(r,\theta,\phi,t) = \Big(f_1(r) + f_3(r)\Big)\left[\cos\chi(3\cos^2\theta - 1) + 3\sin\chi\sin\theta\cos\theta\cos\lambda\right] , \tag{A13}$$

$$E^{\hat{\theta}}(r,\theta,\phi,t) = \frac{1}{2}\Big(f_2(r) + f_4(r)\Big)\left[\cos\chi\sin 2\theta - \sin\chi\cos 2\theta\cos\lambda\right] + \frac{rF(r)}{2N}\left(\Omega - \omega\right)\sin\chi\cos\lambda , \tag{A14}$$

$$E^{\hat{\phi}}(r,\theta,\phi,t) = \frac{1}{2}\left[f_2(r) + f_4(r) + \frac{rF(r)}{N}(\Omega - \omega)\right]\sin\chi\cos\lambda , \tag{A15}$$

where $F(r)$, $G(r)$, and $f_1(r) - f_4(r)$ the radial functions.

$$F(r) = -\frac{3\mu}{4M^3}\left[\ln N^2 + \frac{2M}{r}\left(1 + \frac{M}{r}\right)\right] , \tag{A16}$$

$$G(r) = \frac{3\mu}{4rM^2}N\left[\frac{r}{M}\ln N^2 + \frac{1}{N^2} + 1\right] , \tag{A17}$$

and

$$f_1(r) = \frac{\mu\Omega}{6cR^2}C_1C_2\left[\left(3 - \frac{2r}{M}\right)\ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4\right] , \tag{A18}$$

$$f_2(r) = -\frac{\mu\Omega}{cR^2}C_1C_2N\left[\left(1 - \frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right] , \tag{A19}$$

$$f_3(r) = \frac{15\mu\omega r^3}{16cM^5}\left\{C_3\left[\left(3 - \frac{2r}{M}\right)\ln N^2 + \frac{2M^2}{3r^2} + \frac{2M}{r} - 4\right] + \frac{2M^2}{5r^2}\ln N^2 + \frac{4M^3}{5r^3}\right\} , \tag{A20}$$

$$f_4(r) = -\frac{45\mu\omega r^3}{8cM^5}N\left\{C_3\left[\left(1 - \frac{r}{M}\right)\ln N^2 - 2 - \frac{2M^2}{3r^2N^2}\right] + \frac{4M^4}{15r^4N^2}\right\} . \tag{A21}$$

Keep in mind that these profile functions satisfy the following equations [17]

$$\frac{d}{dr}\left[r^2F(r)\right] + \frac{2r}{N}G(r) = 0 , \tag{A22}$$

$$\frac{d}{dr}\left[r\,N\,G(r)\right] + F(r) = 0 , \tag{A23}$$

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and

$$\frac{d}{dr}\left[r^2\, f_1(r)\right] + \frac{r}{N}\, f_2(r) = 0\,, \tag{A24}$$

$$\frac{d}{dr}\left[r\, N\, f_2(r)\right] + 6\, f_1(r) = 0\,, \tag{A25}$$

$$\frac{d}{dr}\left[r^2\, f_3(r)\right] + \frac{r}{N}\, f_4(r) = 0\,, \tag{A26}$$

$$\frac{d}{dr}\left[r\, N\, f_4(r)\right] + 6\, f_3(r) + 3\omega\, rF(r) = 0\,, \tag{A27}$$

which are very useful equations in the calculation of the component of the vector potential of the electromagnetic fields.

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