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Shadow of rotating Bardeen black hole with clouds of string

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The detailed study of horizon structure and the shadow cast by a rotating Bardeen black hole with a string cloud field. First, we obtained horizon of the spacetime and photon sphere of black hole is also studied for different values of parameters g, a and b. Influence of both parameters a and g orbits are coming to central object and for g is inverse. Later, we have discussed black hole shadow and their observables parameters R_{sh} and δ_s are obtained for various parameters of spacetime metric.

I. INTRODUCTION

Currently most observations and experiments agreed with the standard general relativity (GR). GR is accepted as standard theory of the gravitational interaction. However, current resolutions of these observations do not fully ignore the modifications to GR. As we mentioned above one of the main motivation of introducing the modifications of GR is to resolve fundamental issues of the GR: singularity problem, non-compatibility with quantum field theory and etc.

The gravitational field is main dynamo of the test particles motion around BH or any other compact object. Thus particle dynamics may play a role of preliminary probing the gravitational interaction and its modifications [1, 2]. It is also worth to note that if particle has non-vanishing electric charge/magnetic dipole momentum, then the electromagnetic field may also affect its dynamics [3–6]. In this sense one may have additional degeneracy problem of effects of gravity and electromagnetic field on test particles motion. In this context it is more convienent to consider massless (consequently, chargeless) particles motion around BH.

Moreover, photon motion may be also used to construct new tests of the modified and alternative theories of gravity. The first test of GR has been performed using the effect of gravitational lensing around the Sun observed by solar eclipse in 1919 [7]. Another important consequence of the light propagation in curved spacetime is the appearance of the shadow of the black hole. If the latter is located between the light source and the observer, then due to the light capturing by the central object one may detect the black spot in the sky. This is referred as black hole shadow and first theoretically predicted by Synge [8] and further developed by Luminet [9] and Bardeen [10]. Various authors have studied the shadow of the black hole/compact objects within general relativity and modified/alternative theories of gravity (see, e.g. [11–23]) and also the authors studied the effect of plasma on the BH shadow in [24–26]. One may use the observation data on the shadow of supermassive BHs at the center of M87 [27] and Sgr A* [28] in order to get estimations on parameters of different gravity models [29].

In this work we plan to study the optical properties of spacetime around rotating Bardeen black hole surrounded by cloud of string. The paper is organized as follows: In Sec. II we have studied the horizon structure and the equation of motion of rotating Barden BH with string clouds. Observable BH shadow is considered in Sec. III. Finally, the detailed discussion of the obtained results is presented in Sec. IV. Whole of the work, we used geometric units system that fix the speed of light and the gravitational constant via $G = c = 1$.

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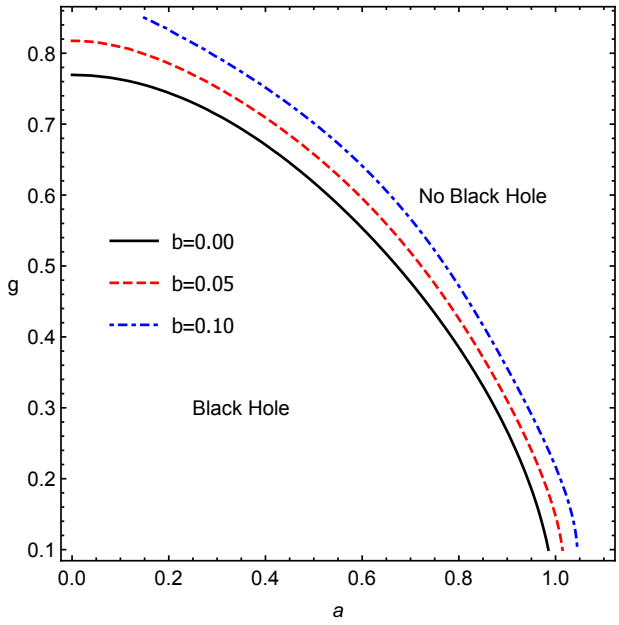


FIG. 1. The separatrix line indicates the border corresponding to extremal black holes which separates black holes from no black holes, with M=1.

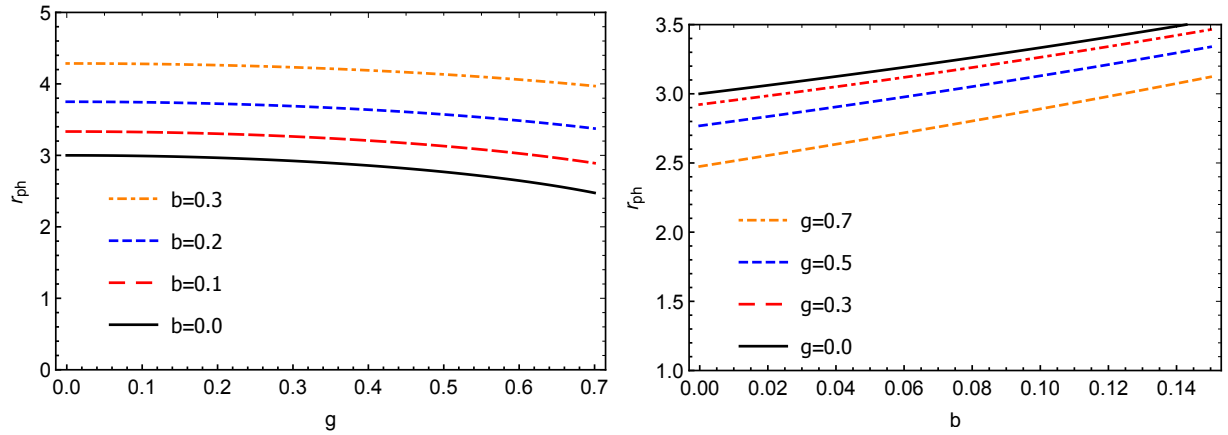


FIG. 2. Plots showing the dependence of photon orbits to the magnetic monopole charge g for selected values of BH parameters b and dependence of photon orbits to the String Clauds parameter b for selected values of BH parameters g with M=1

II. ROTATING BARDEN BH WITH STRING CLOUDS

In the Boyer-Linquist coordinate system, we take into account the spherically symmetric black hole solution reported by the researchers in Ref. [30]

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

with

$$f(r) = 1 - b - \frac{2Mr^2}{\sqrt{(g^2+r^2)^3}}. \tag{2}$$

To discuss the radius of the photon sphere, we use the condition:

$$\frac{d}{dr}\gamma(r)^2\Big|_{r=r_{ph}} = 0 \tag{3}$$

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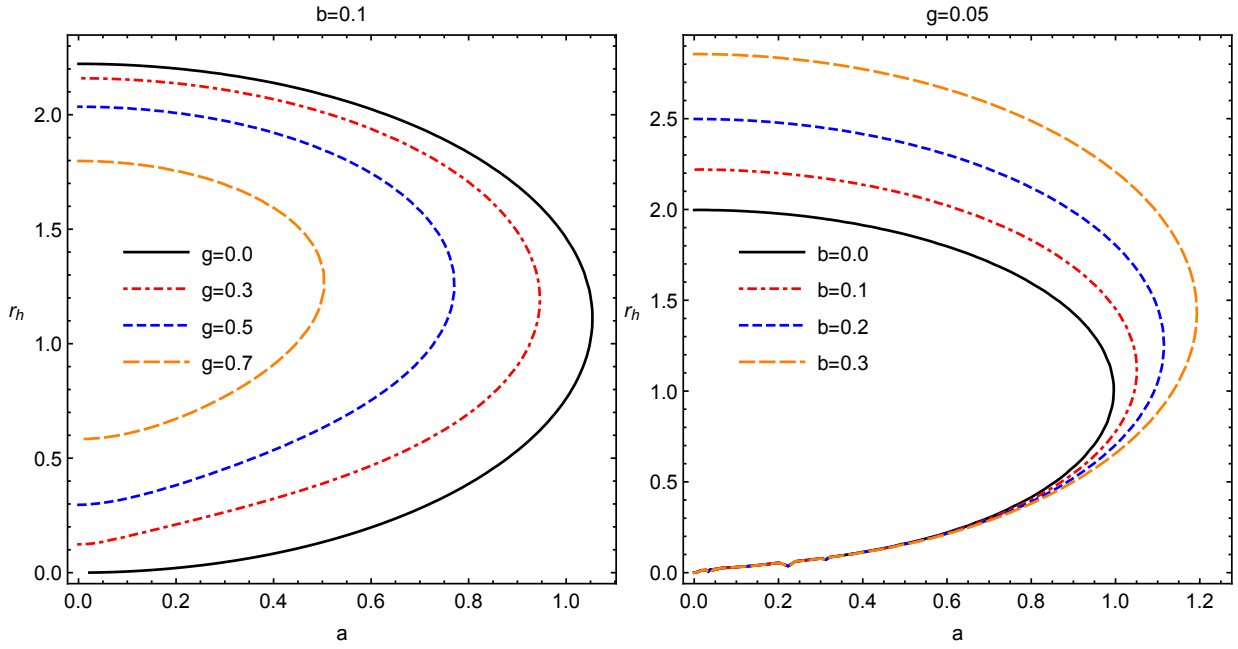


FIG. 3. Variation of horizon radius with increase of BH spin parameter a for the selected values of parameter b and g, with M=1.

with

$$\gamma(r)^2 = \frac{g_{22}}{g_{00}} = \frac{r^2}{f(r)} \tag{4}$$

where g_{00} and g_{22} are elements of metric.

In Fig (2), we have plotted the radius of photon sphere with respect to String Clouds parameter b and Barden parameter g using eq.(3). It can be seen that the outer horizon shifting towards the smaller value of r for the magnetic monopole charge g which implies that the outer horizon shifting towards the central singularity, but with increasing the parameter b the photon radius r_{ph} increases, meaning that BH with string clouds has bigger photon radii than Kerr black hole. Indeed, for a Schwarzschild black hole $r_p = 3M$ and it can be seen clear from the (2) graphic.

Using Newman-Janis algorithm (see. [31, 32]) we can easy get rotating Bardeen BH with string a cloud of string from eqs.(1) and (2). The Rotating Barden black hole with string clouds metric can be presented as

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 \\ &\quad + \frac{\sin^2\theta}{\Sigma}\left(adt - (r^2+a^2)d\phi\right)^2. \end{aligned} \tag{5}$$

For the above metric (5), the metric Σ and Δ are

$$\Sigma = r^2 + a^2\cos^2\theta, \tag{6}$$

$$\Delta = (1-b)r^2 + a^2 - \frac{2Mr^4}{(r^2+g^2)^{3/2}}. \tag{7}$$

where M is the BH mass, a is the spin parameter of the BH, b is String clouds parameter and g is Barden parameter. The above space-time metrics refer to the case of the standard Kerr BH ($b, g \to 0$) and Schwarzschild BH ($b, g \to 0$, and $a \to 0$) in general relativity in the limiting cases. The horizon of the black hole spacetime (5) can be obtained by setting Δ=0. Solving Δ=0 numerically, the behaviour of the radius of the horizon for different values of the parameters b is shown in Fig. (3) for fixed value of Barden parameter g. From the Fig. (3) one may see that the radius of the horizon of Rotating Barden BH with string clouds increases with the increasing value of the parametr b, and therefore, the radius of the horizon of the Kerr black hole is smaller than the radius of the horizon of Rotating Barden BH with string clouds.

Now we analyze properties of horizon structure for Rotating Barden BH with string clouds . Using eq. (7), one may study horizon in detail. Fig. (3) shows the variation of the outer horizon with the change of parameter b. One may see that with the

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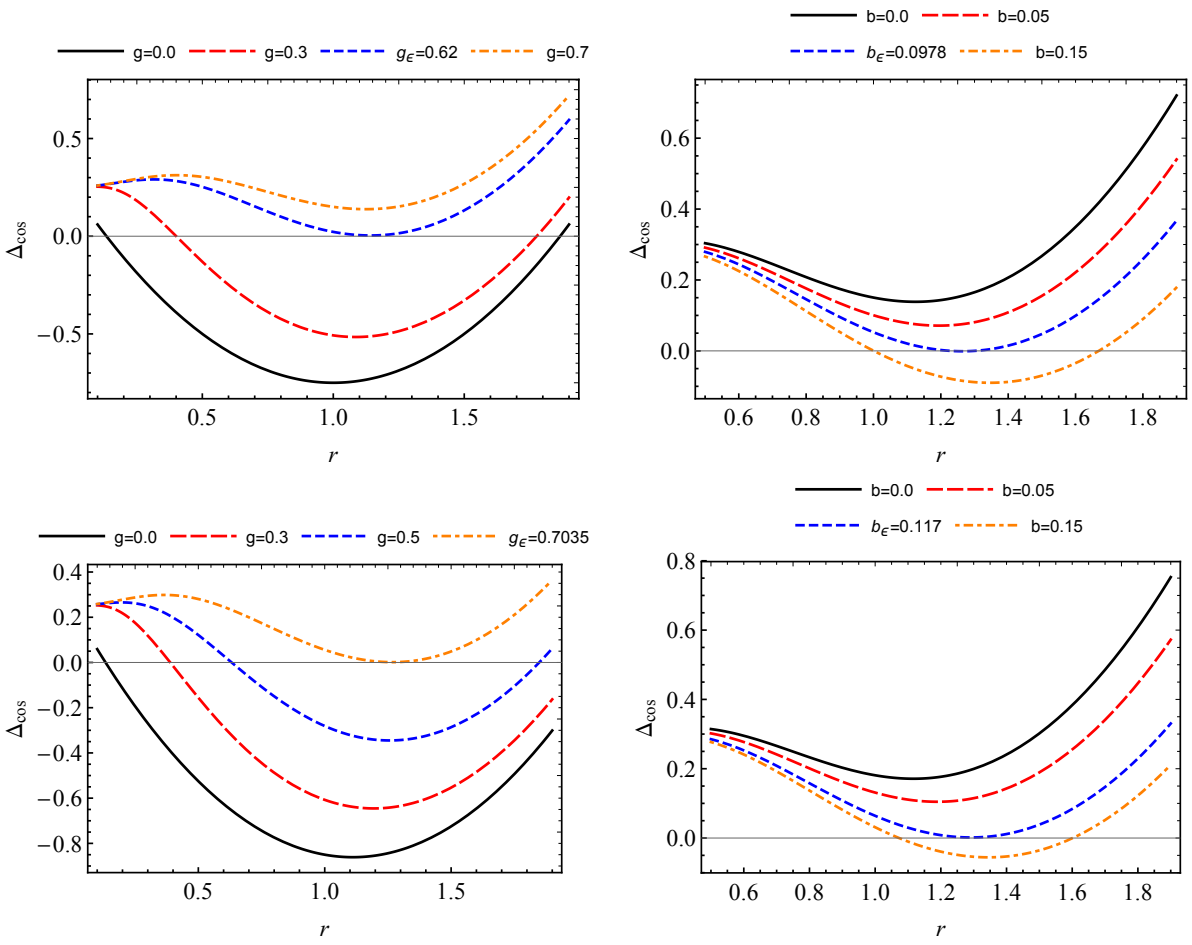


FIG. 4. The dependence of delta function from the radial coordinate for selected values of the BH spin parameter a and parameters $b; g$: $a = 0.5; b = 0$ (top, left panel); $a = 0.5; g = 0.7$ (top, right panel); $a = 0.5; b = 0.1$ (bottom, left) and $a = 0.1; g = 0.72$ (bottom, right panel) with $M = 1$.

increase of parameter b the radius of outer horizon is increased. In addition, we may get the extreme point for the parameter g and spin parameter a for different values of parameter b of the BH as shown in Fig. (1). We have found that there are BH and naked singularity regions and got the dependence of the boundary between them from the BH parameters a and g. The separatrix line indicates the decrease of the maximum value of the extremal BH spin with the increase of the parameter g. One can see that under the effect of the parameter g less value for BH spin is requested in order for an extremal black hole to be transformed into the naked singularity. In Fig. (4) we have shown the dependence of Δ from the radial coordinate. From the obtained dependence one may easily explain the extreme point for various values of BH parameters g and b. Radius of event horizon is decreased with the increase in BH parameters g. Accordingly, the Rotating Barden BH has smaller event horizon radii than the Kerr BH with g parameter. However, radius of event horizon is increased with the increase in BH parameters b. Accordingly, the BH with string clouds has bigger event horizon radii than the Kerr BH.

A. Null Geodesics

In this subsection we investigate massless particle motion around BH described by the rotating BH with string clouds space-time metric (5). The equations of motion properly describe the light rays trajectory around a compact object. The equations of motion of photon can be found using the Hamilton-Jacobi equation ([33]) as

$$\frac{\partial S}{\partial \lambda} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}, \tag{8}$$

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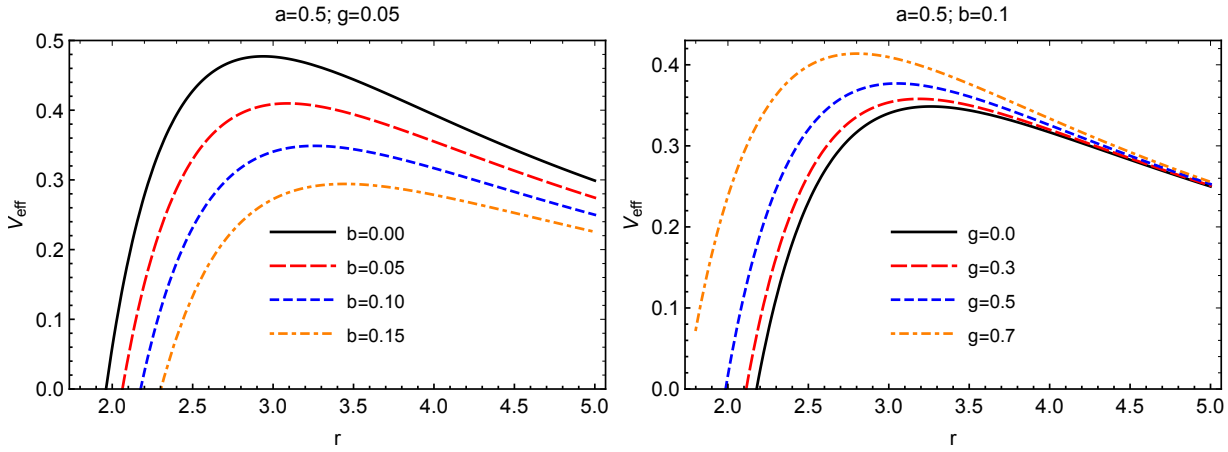


FIG. 5. Effective potential around BH for the massless particle, with M=1.

where λ is an affine parameter and $g^{\mu\nu}$ is the metric tensor. Using method of separation of variables, one can write action S for the photon as

$$S = -Et + L\phi + S_r(r) + S_\theta(\theta), \tag{9}$$

where $E = -g_{t\mu}\dot{x}^\mu$ and $L = g_{\phi\mu}\dot{x}^\mu$ are constants of motion (conserved quantities) and refer to the conserved energy and momentum of the particles, respectively. $S_r(r)$, $S_\theta(\theta)$ in (9) are the function of r and θ, respectively. Using the Eq. (9) and the Eq. (8) one may obtain the geodesic equations in the following form

$$\begin{aligned}
\Sigma\frac{dt}{d\lambda} &= a(L - aE\sin^2\theta) \\
&\quad + \frac{r^2+a^2}{\Delta}\Big(E(r^2+a^2) - aL\Big), &(10)\\
\Sigma\frac{dr}{d\lambda} &= \pm\sqrt{\mathcal{R}}, &(11)\\
\Sigma\frac{d\theta}{d\lambda} &= \pm\sqrt{\Theta}, &(12)\\
\Sigma\frac{d\phi}{d\lambda} &= (L\csc^2\theta - aE) + \frac{a}{\Delta}\Big(E(r^2+a^2) - aL\Big), &(13)
\end{aligned}$$

where $\mathcal{R}$ and Θ have the following forms

$$\begin{aligned}
\mathcal{R} &= \Big[(r^2+a^2)E - aL\Big]^2 - \Delta\Big[\mathcal{K} + (L-aE)^2\Big], &(14)\\
\Theta &= \mathcal{K} + \cos^2\theta\left(a^2E^2 - L^2\sin^{-2}\theta\right), &(15)
\end{aligned}$$

where $\mathcal{K}$ is the Carter constant. Using the expression for $dr/d\lambda$ one may rewrite connection between radial equation of motion and effective potential in the form

$$\left(\Sigma\frac{dr}{d\lambda}\right)^2 + V_{eff} = 0. \tag{16}$$

The effective potential of radial motion in the equatorial plane (where $\theta = \pi/2$) has the following form

$$V_{eff} = \frac{\Delta(\mathcal{K} + (L-aE)^2) - ((a^2+r^2)E - aL)^2}{2r^4}. \tag{17}$$

Using the expression (17) one may plot the graphs of the dependence of the effective potential from the radial coordinate. Fig. (5) shows the radial dependence of the effective potential of radial motion of photons for the selected values of g an b. It is the general property of effective potential with respect to circular orbits (or photon orbits) for different values of spacetime metric. One can see that a peak of the effective potential corresponding to photon orbit is increased and shifted towards the left

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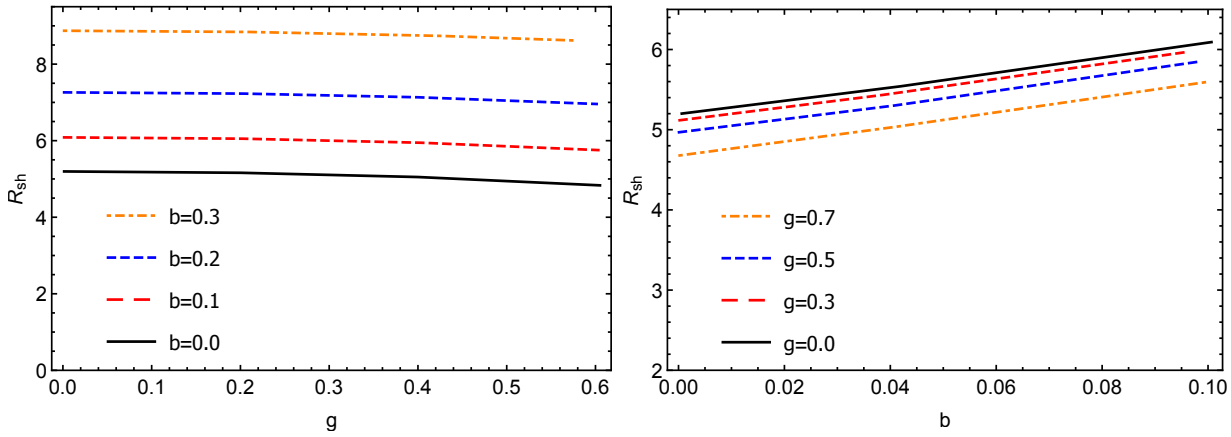


FIG. 6. Variation of the radius R_s of the black hole shadow withe the parameters g and b with M=1

direction with the increase of the value of parameter g. However a peak of the effective potential corresponding to photon orbit is decreased and shifted towards the right direction with the increase of the value of parameter b. It can be seen from the plots of the effective potential that the photon orbits come more closer towards the central object with an increase in the magnetic monopole charge g, but the photon orbits get more away from the central object with an increase in the parameter b in the rotating black hole with string clouds spacetime metric.

III. BLACK HOLE SHADOW

A. Nonrotating black hole shadows

Using Synge method we can study shadow of black hole for the non-rotating case[8]:

$$\sin^2 X_s = \frac{\gamma(r_{ph})^2}{\gamma(r_0)^2} \tag{18}$$

where X_s is the angular radius of the black hole shadow and r_0 is the observe position and r_{ph} is the photon sphere.
For the large distance observation Eq. (18) will be as

$$\sin^2 X_s = \frac{r_{ph}^2}{f(r_{ph})} \frac{f(r_0)}{r_0^2} \tag{19}$$

And now one can find linear radius of BH shadow for large distance observation using Eq. (19) as

$$R_{sh} = \frac{r_{ph}}{\sqrt{f(r_{ph})}} \tag{20}$$

Finally Eq. (20) can explain shadow of BH for the non-rotating case.
From the Fig (6) we can see that with the increase of String clouds parameters b the size of BH shadow increases. Contrastingly, with the increase of the magnetic monopole charge g the radius of BH shadow decreases.

B. rotating black hole shadows

In order to study the shadow of the rotating Barden BH with string clouds field one may introduce the following impact parameters $\xi = L/E$ and $\eta = \mathcal{K}/E^2$ and obtain the following equation for $R(r)$ as

$$\mathcal{R} = \Big[(r^2 + a^2) - a\xi\Big]^2 - \Delta\Big[\eta + (\xi - a)^2\Big]. \tag{21}$$

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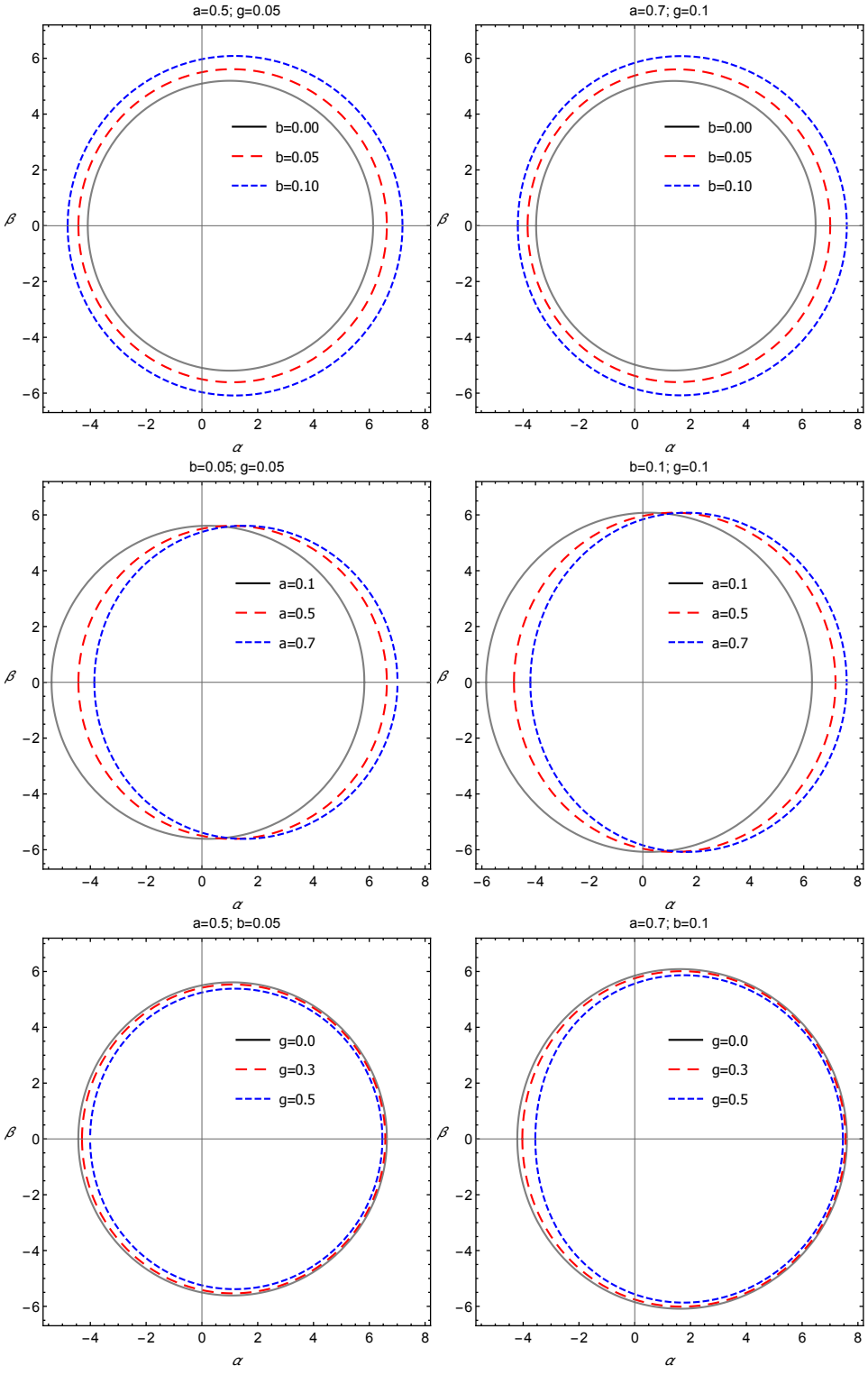


FIG. 7. Shadow of the rotating Barden black hole with string clouds for selected values of spin parameter a and magnetic monopole charge g at fixed values for BH parameter b, for selected values of a and b at fixed g and for selected values of parameter b and magnetic monopole charge g at fixed values for BH parameter a, with M=1.

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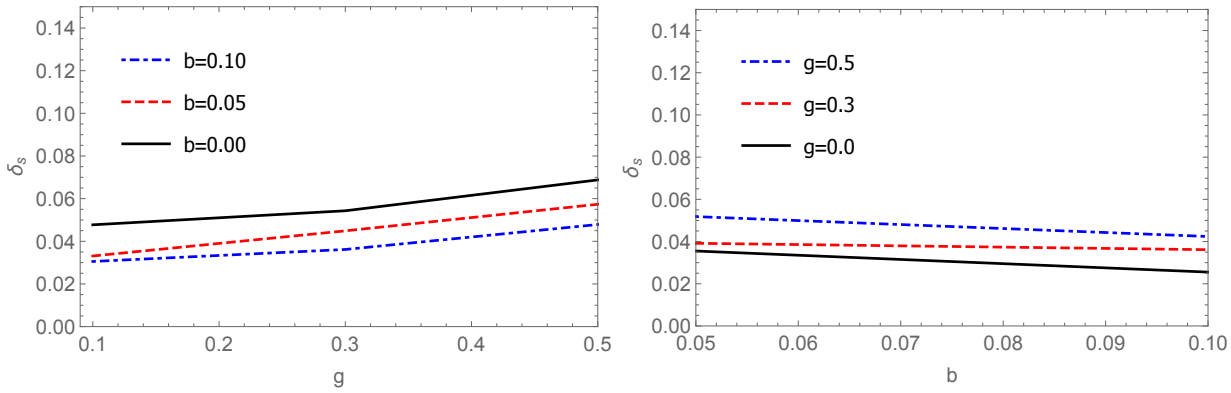


FIG. 8. Plots of distortion for different values b and g. Here we have taken a=0.5 and M=1

The following conditions should be satisfied in order to have unstable circular orbits of photons

$$\mathcal{R} = 0, \quad \frac{\partial \mathcal{R}}{\partial r} = 0, \quad \frac{\partial^2 \mathcal{R}}{\partial r^2} < 0, \tag{22}$$

where the parameters ξ and η describe the borders of the BH shadow. Using Eqs. (21) and (22) one can get ξ and η as

$$\xi = \frac{\left(a^2 + r^2\right)\Delta' - 4\Delta r}{a\Delta'}, \tag{23}$$

$$\eta = \frac{r^2\left(16\Delta\left(a^2 - \Delta\right) - r^2\Delta'^2 + 8\Delta r\Delta'\right)}{a^2\Delta'^2}, \tag{24}$$

The shadow of the rotating BH with string clouds is described by the null geodesics represented in celestial coordinates α and β:

$$\alpha = \lim_{r\to\infty}\left(-r^2\sin\theta\frac{d\phi}{dr}\bigg|_{\theta\to\theta_0}\right) = -\xi\csc\theta_0, \tag{25}$$

$$\beta = \lim_{r\to\infty}\left(r^2\frac{d\theta}{dr}\bigg|_{\theta\to\theta_0}\right) = \pm\sqrt{\eta + a^2\cos^2\theta_0 - \xi^2\cot^2\theta_0}. \tag{26}$$

Here, we assumed the observer is far away from the black hole ($r \to \infty$) and θ_0 is the angel between the line of the sight and the spin axes of black hole, namely, the inclination angle. Since the black hole spacetime is asymptotically flat, we can consider a static observer at an arbitrarily large distance. We analyze BH shadow in an equatorial plane ($\theta_0 = \pi/2$) and so we can write

$$\alpha = -\xi \tag{27}$$

$$\beta = \pm\sqrt{\eta} \tag{28}$$

In order to obtain the silhouette of the BH shadow one may plot the line described by the ξ vs η in the celestial coordinates. Fig 7 shows the shadows of the BH for selected values of the BH spin parameter a, parameters b and g of rotating Barden black hole with string clouds. It can be seen from the results (see Fig 7) that the size of the BH shadow decreased with increasing the magnetic monopole charge but the shadow of BH increased with increasing parameter b of the rotating Barden BH with string clouds. Under influence of BH spin parameter a, pure circle for the border of BH shadow is violated.

For axial symmetric black hole, the shadow is distorted and Hioki and Maedia [12] have first introduced two parameters as radius for the size of BH shadow and deflection to fully describe them. Considering the shape of the shadow of the BH is a circle as shown in Fig. 7 (See reference [12]). Let us consider that shadow as a circle is passing through (A) (top), (B) (bottom), and the most left end C of its boundary. The coordinates to these points be (α_t, β_t), (α_b, β_b) and $(\alpha_r, 0)$ respectively. The radius R_s of the circle defines the radius of shadow. The point C on the circle represents the unstable retrograde circular orbit for an observer on the equatorial plane. To extract detailed information from the shadow and connect astronomical observation, we find R_s radius of BH shadow and distortion (deviation) parameter δ_s. The distortion parameter δ_s is defined as

$$\delta_s = \frac{D_{cs}}{R_{sh}}, \tag{29}$$

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here D_{cs} is the difference between the right endpoints of the shadow. The observable R_s is defined as

$$R_s = \frac{(\alpha_t - \alpha_r)^2 + \beta_t^2}{2(\alpha_t - \alpha_r)}. \tag{30}$$

To further see the effect of all parameters on the shape of the shadow for rotating Barden BH with string clouds, we have plotted the distortion with respect to the magnetic monolope charge g and String clouds parameters b by obtain it numerically using the shadow plots Fig. 7 for different values g and b From the Fig 8 the dependence of δ_s from String clouds parameters b shows that with the increase of parameter b BH shadow get the shape of circle rather than ellipse. However, with the increase of magnetic monopole charge g the shape is distorted. If the rotation parameter a increases, the shape of BH is chipped away. The distortion of the shape of the shadow can be understood as follows: When a black hole has a spin, the radius of the direct circular orbit decreases and then the left endpoint of the shadow moves to the right, while the retrograde's one increases and then the right endpoint moves to the right as well.

IV. CONCLUSIONS AND DISCUSSIONS

In this work we have studied in detail the shadow of the rotating Bardeen BH with a cloud of string described by metric elements (5). The obtained results can be summarized based on the following statements:

- The structure of the rotating BH horizon in the presence of the additional the magnetic monopole charge parameter g and parameter b is studied. Radius of horizon of the BH is increased under effect of the parameter b which is presented in Fig.3.
- The orbits of photons in rotating Barden BH with string clouds spacetime have been analyzed using the effective potential. It has been shown that with the increase the String clouds parameter b of the BH in rotating Barden BH the circular orbits of photon around BH move away from the central object and with the increase of parameter of BH g the circular orbits of photon around BH shift toward the central object.
- The equation of motion of the photon around BH in spacetime metric is obtained analytically using the Hamilton-Jacobi equation and the separability method. The shadow of BH in celestial coordinates is obtained from the geodesic equations (see Fig.7).
- In addition, we have also studied the effect of all parameters to the shape of rotating Barden BH with string clouds and analyzed distortion parameters δ_s and plotted radius of shadow R_s for non-rotating BH.

[1] A. Jawad, F. Ali, M. Jamil, and U. Debnath, Communications in Theoretical Physics **66**, 509 (2016), arXiv:1610.07411 [gr-qc].
[2] M. Bañados, J. Silk, and S. M. West, Physical Review Letters **103**, 111102 (2009).
[3] R. M. Wald, Phys. Rev. D. **10**, 1680 (1974).
[4] A. N. Aliev and N. Özdemir, Mon. Not. R. Astron. Soc. **336**, 241 (2002), gr-qc/0208025.
[5] A. Abdujabbarov and B. Ahmedov, Phys. Rev. D **81**, 044022 (2010), arXiv:0905.2730 [gr-qc].
[6] F. de Felice, F. Sorge, and S. Zilio, Classical and Quantum Gravity **21**, 961 (2004).
[7] F. W. Dyson, A. S. Eddington, and C. Davidson, Philosophical Transactions of the Royal Society of London Series A **220**, 291 (1920).
[8] J. L. Synge, Mon. Not. R. Astron. Soc. **131**, 463 (1966).
[9] J. P. Luminet, Astron.Astrophys. **75**, 228 (1979).
[10] J. M. Bardeen, in *Black Holes (Les Astres Occlus)* (1973) pp. 215–239.
[11] H. Falcke, F. Melia, and E. Agol, Astrophys. J. **528**, L13 (2000), arXiv:astro-ph/9912263 [astro-ph].
[12] K. Hioki and K.-I. Maeda, Phys. Rev. D **80**, 024042 (2009).
[13] A. Abdujabbarov, F. Atamurotov, Y. Kucukakca, B. Ahmedov, and U. Camci, Astrophys. Space. Sci. **344**, 429 (2013), arXiv:1212.4949 [physics.gen-ph].
[14] L. Amarilla and E. F. Eiroa, Phys. Rev. D **85**, 064019 (2012).
[15] A. Abdujabbarov, M. Amir, B. Ahmedov, and S. G. Ghosh, Phys. Rev. D **93**, 104004 (2016), arXiv:1604.03809 [gr-qc].
[16] N. Tsukamoto, Phys. Rev. D. **97**, 064021 (2018), arXiv:1708.07427 [gr-qc].
[17] A. A. Abdujabbarov, L. Rezzolla, and B. J. Ahmedov, Mon. Not. R. Astron. Soc. **454**, 2423 (2015), arXiv:1503.09054 [gr-qc].
[18] F. Atamurotov, U. Papnoi, and K. Jusufi, Class. Quant. Grav. **39**, 025014 (2022), arXiv:2104.14898 [gr-qc].
[19] F. Sarikulov, F. Atamurotov, A. Abdujabbarov, and B. Ahmedov, Eur. Phys. J. C **82**, 771 (2022).
[20] U. Papnoi and F. Atamurotov, Physics of the Dark Universe **35**, 100916 (2022), arXiv:2111.15523 [gr-qc].
[21] S.-W. Wei and Y.-X. Liu, J. Cosmol. A. P. **2013**, 063 (2013), arXiv:1311.4251 [gr-qc].

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[22] M. Afrin, R. Kumar, and S. G. Ghosh, Mon. Not. Roy. Astron. Soc. **504**, 5927 (2021), arXiv:2103.11417 [gr-qc].
[23] M. Khodadi, Nucl. Phys. B **985**, 116014 (2022), arXiv:2211.00300 [gr-qc].
[24] V. Perlick, O. Y. Tsupko, and G. S. Bisnovatyi-Kogan, Phys. Rev. D. **92**, 104031 (2015), arXiv:1507.04217 [gr-qc].
[25] V. Perlick and O. Y. Tsupko, Phys. Rev. D. **95**, 104003 (2017), arXiv:1702.08768 [gr-qc].
[26] F. Atamurotov, K. Jusufi, M. Jamil, A. Abdujabbarov, and M. Azreg-Ainou, Phys. Rev. D **104**, 064053 (2021), arXiv:2109.08150 [gr-qc].
[27] K. Akiyama and et al. (Event Horizon Telescope Collaboration), Astrophys. J. **875**, L1 (2019), arXiv:1906.11238 [astro-ph.GA].
[28] K. Akiyama and et al., Astrophys. J. Lett **930**, L12 (2022).
[29] S. Vagnozzi and et al., arXiv e-prints , arXiv:2205.07787 (2022), arXiv:2205.07787 [gr-qc].
[30] M. E. Rodrigues and H. A. Vieira, Phys. Rev. D **106**, 084015 (2022), arXiv:2210.06531 [gr-qc].
[31] M. Azreg-Aïnou, Phys. Rev. D **90**, 064041 (2014), arXiv:1405.2569 [gr-qc].
[32] E. T. Newman and A. I. Janis, Journal of Mathematical Physics **6**, 915 (1965).
[33] B. Carter, Phys. Rev. **174**, 1559 (1968).

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