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Electromagnetic field due to uniformly charged ring and current ring

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The main purpose of the paper is to explore the electrostatic potential and electric field due to the axially-symmetric uniformly charged ring at any point of the space. We first show the exact analytical solution of the Poisson equation for the electrostatic potential due to a charged ring and find the electric field. The explicit for of the Poisson equations for vector potential of the electromagnetic fields in the spherical coordinates has been derived. It is shown that obtained Poisson equations is not for single component of the vector potential but the combination of them. As a result, one of the particular solution for the vector potential generated by the axially symmetric, static current ring has been found. It has been produced an exact analytical solution for electromagnetic fields in alternative way directly solving Maxwell equations for the electric and magnetic fields which coincides with previous results. Finally, it has shown that obtained results for the vector potential can be expressed in terms of the complete elliptic integrals of the first and second kind. It is shown that electric field due to uniformly charged ring and magnetic field due to current ring can be described by well-known the elliptic integrals of the first and second kind. It is also shown that obtained results for the electrostatic potential and electric field is applicable for gravitational potential and gravitational acceleration due to massive ring.

I. INTRODUCTION

The electrodynamics theory is one of the well investigated and formulated relativistic theory that describes electromagnetic phenomena which is applicable in different areas of the physics. This theory is governed by well-know Maxwell equations:

$$\nabla \cdot \mathbf{E} = 4\pi\rho \ , \qquad \nabla \times \mathbf{B} - \partial_t \mathbf{E} = 4\pi \mathbf{J} \ , \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0 \ , \qquad \nabla \times \mathbf{E} + \partial_t \mathbf{B} = 0 \ , \tag{2}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields, ρ and $\mathbf{J}$ are the charge density and current, respectively. The electromagnetic wave equations can be obtained by setting source terms to equal to zero, i.e. $\rho = \mathbf{J} = 0$. However, if these fields are stationary (time independent), then the electrodynamics theory separates into classes, namely, electrostatics and magnetostatics theories. The electrostatic theory describes by equations $\nabla \cdot \mathbf{E} = 4\pi\rho$ and $\nabla \times \mathbf{E} = 0$, while magnetostatics is described by the equations, $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 4\pi\mathbf{J}$.

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The electric field due to the axially-symmetric uniformly charged ring is quite common problem in classical electrostatics. This problem is solved long time ago and presented in many textbooks in electrodynamics and electrostatics[1–13], however, these solutions are given by in the form of series which might inconvenience for the highest power of series. From this point of view, it is also interesting to consider this problem in different approach which can be easily understandable and applicable in different areas of physics. One must point out that in Ref. [14], the same problem has been discussed, and the solution has presented in the form of integrals, however, the charged distribution along the ring has been incorrectly considered. In Ref. [15], the integral form of the potential is shown without any explanations. Recently, in [16] it has been investigated two different ways of finding the solutions of electrostatic problems including the charged ring's problem. The electrostatic potential and electric field due to a uniformly charged disk has been studied in [17].

The magnetic field due to the electric current ring is one of the fundamental problem in electrodynamics. The analytical expression for the magnetic field due to current ring has been discussed in the literatures, see for example [1–13, 15]. The exact analytical expressions for the magnetic field around current loop in Cartesian, spherical and cylindrical coordinates has been discussed in [18]. In the textbook by Jackson, it is shown that magnetic field of the current loop can be expressed in terms of the complete elliptic integrals. However, above references the derivations of the equations the vector potential or magnetic field of the current loop may confuse the reader to understand what is behind

From these point of view in the present paper, we aim to show in very detailed derivations of the equations for the magnetic field due to axially symmetric and static electric current ring at arbitrary point of the space. Assuming the thickness of the ring is negligibly smaller than it's radius and electric current is stationary along the ring. We believe that the results in present paper will be interesting and useful to people involved in teaching electrodynamics courses as well as in research work in the different area physics related to the electromagnetism.

In the present paper, we are aiming to investigate the electrostatic potential and electric field due to charged ring at arbitrary point of the space. Assuming the thickness of the ring is negligibly smaller than it's radius and electric charge is uniformly distributed along the ring.

The paper is organized as follows. In section II, we briefly describe principle properties of the D'Alembert equationsthat governs electromagnetic and gravitational phenomena. In section III, we present the exact analytical expressions for the electrostatic potential and the electric field due to the axially-symmetric charged ring. In section IV, we present the exact analytical expressions for the vector potential as well as for the magnetic field due to the axially-symmetric current ring. Section V is devoted to investigate the same problem in different approach by directly solving Maxwell equations. In section VI, we discuss obtained results. Finally, we summarize the obtained results in section VII.

II. D'ALEMBERT EQUATIONS

The fundamental D'Alembert equations for the vector potential of the electromagnetic fields, i.e. $A_\alpha = (\Phi, \mathbf{A})$ with given source, i.e. $J^\alpha = (\rho, \mathbf{J})$, are given as [11, 12]

$$\Delta\Phi - \partial_t^2\Phi = -4\pi\rho \ , \qquad \Delta\mathbf{A} - \partial_t^2\mathbf{A} = -4\pi\mathbf{J} \ . \tag{3}$$

where Δ is Laplacian operator, ∂_t^2 is the second order time derivative operator. The solutions of these equations can be expressed as [11]

$$\begin{aligned} \Phi(t,\mathbf{r}) &= \int \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\rho(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|) \ , \\ \mathbf{A}(t,\mathbf{r}) &= \int \frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\mathbf{J}(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|) \ , \end{aligned} \tag{4}$$

so-called retarded potential for moving charged particle. Here $\mathbf{r}$ is radius vector where field measures and $\mathbf{r}'$ is a position of the source. For the stationary fields, equations (3) reduce the Poisson equations for the time and spatial components for the vector potential. The solution of Poisson equation can be expressed in the form of Green's function as expressed in solutions (4).

III. ELECTROSTATIC POTENTIAL AND ELECTRIC FIELD DUE TO CHARGED RING

We now focus on the electrostatic potential and electric field due to a uniformly charged ring. The general form of the Poisson equation is given by [11, 12]

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$$\Delta\Phi(\mathbf{r}) = -4\pi\rho(\mathbf{r}) \ , \tag{5}$$

where $\Phi(\mathbf{r})$ and $\rho(\mathbf{r})$ are, scalar potential and the source, functions of radius vector $\mathbf{r} = (x, y, z)$, respectively. The Laplacian operator Δ is defined as, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$. The solution of the equation (5) can be expressed in terms of Green's function as [11, 12]

$$\Phi(\mathbf{r}) = -\int d\mathbf{r}' G(\mathbf{r}, \mathbf{r}')\rho(\mathbf{r}') = \int \frac{d\mathbf{r}'\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \ , \tag{6}$$

where $\mathbf{r}$ is a position where potential measures, $\mathbf{r}'$ is a position of the source and $d\mathbf{r}' = dx'dy'dz'$ is the volume element. So far we consider everything in Cartesian coordinates, however, the spherical coordinates, (r, θ, ϕ), are suit to consider the electrostatic potential or electric field around the uniformly charged ring. Before facing the solution (6) of the Poisson equation, we recall basic calculations in spherical coordinates. So that the volume element takes a form $d\mathbf{r}' = r'^2 \sin\theta' dr' d\theta' d\phi'$ and the Green's function is defined as (See, A for detials)

$$G(\mathbf{r}, \mathbf{r}') = -\frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\frac{1}{\sqrt{r^2 + r'^2 - 2rr'\left[\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')\right]}} \ , \tag{7}$$

as well as the Laplacian operator in the spherical coordinates can be found as

$$\Delta = \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{r^2}\Delta_\Omega \ , \qquad \Delta_\Omega = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\phi^2 \ . \tag{8}$$

Now, we focus on the sourse term in the Poisson equation. The charge density of the ring of a radius a and total charge q is given by [12]

$$\rho(\mathbf{r}) = \frac{q}{2\pi a^2}\delta(r - a)\delta(\cos\theta) \ , \tag{9}$$

which is independent of angle ϕ. It turns out that the potential also should be independent of angle ϕ due to an asymmetry. However, keep in mind that integration over angle ϕ' should be included when we evaluate integral over the volume element in the spherical coordinate, because the Green's function depends on it.

Taking into account all the facts above and using the expressions (7) and (9), the solution Poisson equation for the electrostatic potential (6) for the uniformly charged ring can be rewritten as

$$\begin{aligned}
\Phi(r, \theta) &= \frac{q}{2\pi a^2}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \int_0^{\infty} dr' \frac{r'^2 \sin\theta'\delta(r' - a)\delta(\cos\theta')}{\sqrt{r^2 + r'^2 - 2rr'(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi')}} \\
&= \frac{q}{2\pi}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \frac{\sin\theta'\delta(\cos\theta')}{\sqrt{r^2 + a^2 - 2ra(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi')}} \\
&= \frac{q}{2\pi}\int_0^{2\pi} \frac{d\phi'}{\sqrt{r^2 + a^2 - 2ra\sin\theta\cos\phi'}} \ .
\end{aligned} \tag{10}$$

Here, we have used the following property of the delta function, $\int dx f(x)\delta(x - x_0) = f(x_0)$, when we evaluate integrals with respect to coordinates r' and θ', respectively. The next task is to describe the expression (10) and explore it in terms of well-known function. Note that the resulting expression should be a function coordinates (r, θ). Indeed, straightforward evaluation of the integral (10) is not a simple task. However, it can be expressed in the form of some special integrals so-called the elliptic integrals. Hereafter performing the simple trigonometry manipulation i.e. $\cos\phi' = 1 - 2\sin^2\phi'/2$, the equation (10) can be rewritten as

$$\Phi(r, \theta) = \frac{q}{2\pi}\int_0^{2\pi} \frac{d\phi'}{\sqrt{r^2 + a^2 - 2ra\sin\theta + 4ra\sin\theta\sin^2\phi'/2}} \ . \tag{11}$$

Now, introducing new variable $\chi = \phi'/2$ with a range $0 \leq \chi \leq \pi$, the expression (11) reads

$$\Phi(r, \theta) = \frac{q}{\pi}\int_0^{\pi} \frac{d\chi}{\sqrt{r^2 + a^2 - 2ra\sin\theta + 4ra\sin\theta\sin^2\chi}} \ . \tag{12}$$

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Finally, hereafter performing simple algebraic manipulations (See, for details B), the expression for the electrostatic potential takes a form:

$$\Phi(r,\theta) = \frac{q}{\pi}\left[\frac{K\left(k_+^2\right)}{r_+} + \frac{K\left(k_-^2\right)}{r_-}\right] , \tag{13}$$

where $r_\pm = \sqrt{r^2 + a^2 \pm 2ra\sin\theta}$ and $k_\pm^2 = \pm 4ra\sin\theta/r_\pm$. Consequently, the non-zero components of the electric field read as

$$E_{\hat{r}}(r,\theta) = -\frac{q}{\pi r}\left[\frac{K\left(k_+^2\right)}{r_+} + \frac{K\left(k_-^2\right)}{r_-}\right] - \frac{q(r^2-a^2)}{\pi r r_+ r_-}\left[\frac{E\left(k_+^2\right)}{r_-} + \frac{E\left(k_-^2\right)}{r_+}\right] , \tag{14}$$

$$E_{\hat{\theta}}(r,\theta) = -\frac{q\cot\theta}{\pi r}\left[\frac{K\left(k_+^2\right)}{r_+} + \frac{K\left(k_-^2\right)}{r_-}\right] + \frac{q\left(r^2+a^2\right)\cot\theta}{\pi r r_+ r_-}\left[\frac{E\left(k_+^2\right)}{r_-} + \frac{E\left(k_-^2\right)}{r_+}\right] , \tag{15}$$

where $K(k)$ and $E(k)$ are, respectively, the complete elliptic integrals of the first and second kind defined as [19]

$$K(k) = \int_0^{\pi/2} \frac{d\chi}{\sqrt{1-k^2\sin^2\chi}} , \qquad K(0) = \pi/2 , \tag{16}$$

$$E(k) = \int_0^{\pi/2} d\chi\sqrt{1-k^2\sin^2\chi} , \qquad E(0) = \pi/2 . \tag{17}$$

One has to emphasize that the expressions (14), (15) represent the electric field in any points around the charged ring, as well as the electrostatic potential in equation (13). These three expressions are main results of the present paper. The electrostatic potential satisfies the following symmetric properties:

$$\Phi(r,\theta) = \Phi(-r,-\theta) , \quad \Phi(r,\theta) = \Phi(-r,\theta) , \quad \Phi(r,\theta) = \Phi(r,-\theta) , \tag{18}$$

In order to make sure about our solutions, we can check them in some special cases. One of the simple cases to determine the electric field in the center of the ring, i.e. $r = 0$. The electrostatic potential and electric field take the form:

$$\Phi(0,\theta) = \frac{q}{a} , \qquad E_{\hat{r}} = E_{\hat{\theta}} = 0 , \tag{19}$$

while along the symmetry axe, i.e. $\theta = 0$, one can get

$$\Phi(r,0) = \frac{q}{\sqrt{r^2+a^2}} , \qquad E_{\hat{r}} = \frac{qr}{\sqrt{(r^2+a^2)^3}} , \qquad E_{\hat{\theta}} = 0 . \tag{20}$$

which coincide with the results thus previously obtained in [13]. In similar way, we can show the asymptotic behavior of the potential and field at the limiting case of $r \to \infty$, one can obtain $\Phi = q/r$, $E_{\hat{r}} = q/r^2$ and $E_{\hat{\theta}} = 0$ which represents results for point-like charge.

In the plane of the ring, i.e. $\theta = \pi/2$, the tangential electric field vanishes, because of $\cot(\pi/2) = 0$, while the radial component takes a form:

$$E_{\hat{r}}(r,\pi/2) = -\frac{q}{\pi r}\left[\frac{K\left(\frac{4ra}{(r+a)^2}\right)}{|r+a|} + \frac{K\left(\frac{-4ra}{(r+a)^2}\right)}{|r-a|}\right] - \frac{q(r-a)}{\pi r|r-a|}\left[\frac{E\left(\frac{4ra}{(r+a)^2}\right)}{|r-a|} + \frac{E\left(\frac{-4ra}{(r+a)^2}\right)}{|r+a|}\right] , \tag{21}$$

as well as the electrostatic potential reads

$$\Phi(r,\pi/2) = \frac{q}{\pi}\left[\frac{K\left(\frac{4ra}{(r+a)^2}\right)}{|r+a|} + \frac{K\left(\frac{-4ra}{(r-a)^2}\right)}{|r-a|}\right] . \tag{22}$$

Fig. 1 draws radial dependence of the electrostatic potential in the plane of the ring for the different radius of the ring in the left panel and in the right panel the same plot is presented but for the dimensionless potential and radial coordinate. Indeed, according to plots, the potential goes to infinity at the distance of the ring's radius. In the center of the ring it is, $\Phi_0 = q/a$, while outside of the ring it falls as similar to $1/r$.

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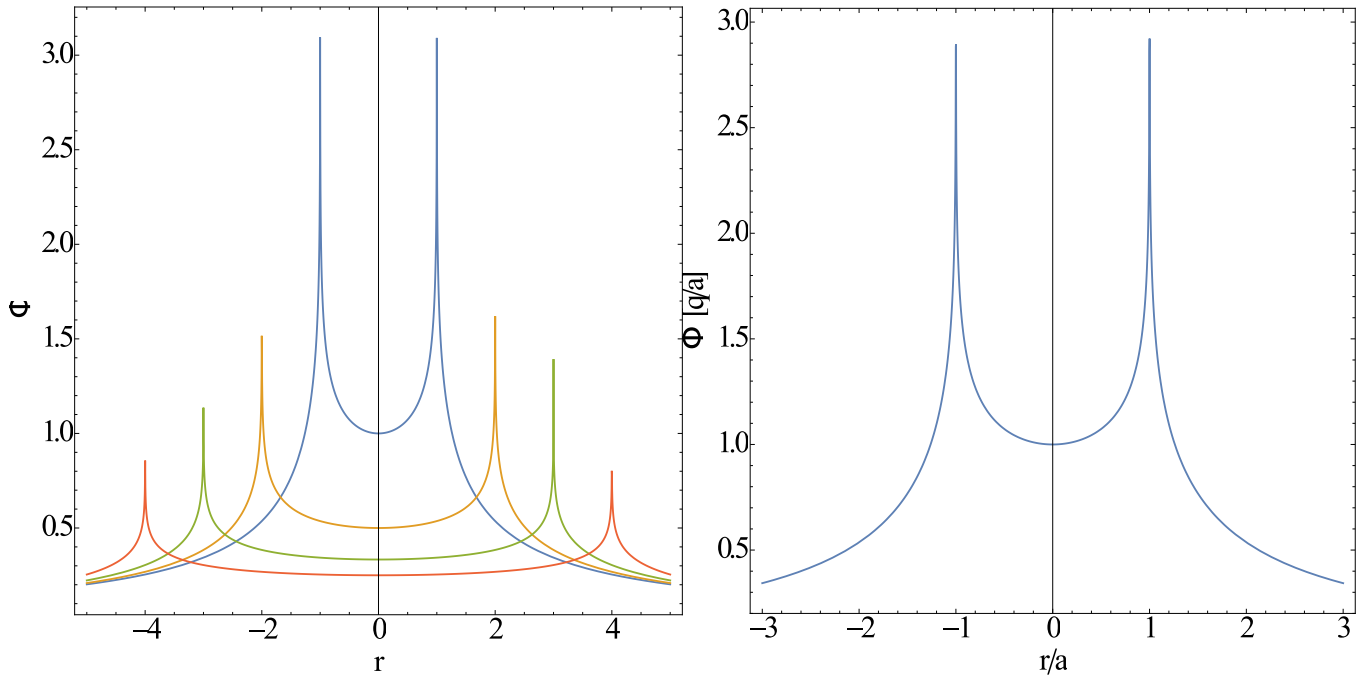


FIG. 1: Radial dependence of the electrostatic potential in the ring's plane.

IV. MAGTETOSTATIC POTENTIAL AND MAGNETIC FIELD DUE TO CURRENT RING

In this section we study the magnetostatic and magnetic field due to a static current ring. In this case, D'Alembert equation in (3) for vector potential $\mathbf{A}$ reduces to the Poisson equation: $\Delta\mathbf{A} = -4\pi\mathbf{J}$. The solution of this equation can be found as [12]

$$\mathbf{A}(t,\mathbf{r}) = \int \frac{d\mathbf{r}'\mathbf{J}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \ . \tag{23}$$

Here $\mathbf{r}$ is a radius vector where field measures and $\mathbf{r}'$ is a position of the source $\mathbf{J}(\mathbf{r}')$. In the spherical coordinates (r,θ,ϕ), the Poisson equations for the components of the vector potential $\mathbf{A} = (A_{\hat{r}}, A_{\hat{\theta}}, A_{\hat{\phi}})$ can be written as [19]

$$\Delta A_{\hat{r}} - \frac{2}{r^2}A_{\hat{r}} - \frac{2}{r^2\sin\theta}\partial_\theta(\sin\theta A_{\hat{\theta}}) - \frac{2}{r^2\sin\theta}\partial_\phi A_{\hat{\phi}} = -4\pi J_{\hat{r}} \ , \tag{24}$$

$$\Delta A_{\hat{\theta}} - \frac{A_{\hat{\theta}}}{r^2\sin^2\theta} + \frac{2}{r^2}\partial_\theta A_{\hat{r}} - \frac{2\cos\theta}{r^2\sin^2\theta}\partial_\phi A_{\hat{\phi}} = -4\pi J_{\hat{\theta}} \ , \tag{25}$$

$$\Delta A_{\hat{\phi}} - \frac{A_{\hat{\phi}}}{r^2\sin^2\theta} + \frac{2}{r^2\sin\theta}\partial_\phi A_{\hat{r}} + \frac{2\cos\theta}{r^2\sin^2\theta}\partial_\phi A_{\hat{\theta}} = -4\pi J_{\hat{\phi}} \ . \tag{26}$$

If we have look at equations (24)-(26), they are coupled equations for the components of the vector potential and there are huge differences between the second equation of (5). Indeed, a question arises whether equation (5) is valid in an arbitrary coordinates of system or is it valid only in Cartesian coordinates?

Now we concentrate on writing the equations (24)-(26) in such way that in the left hand sides of equations, the operator Δ should stands and acts on the vector field. By multiplying $\cos\theta$ onto both sides of equation (24) and $\sin\theta$ onto both sides of equation (25), subtract the final equations and after performing simple algebraic manipulations, we obtain

$$\Delta(A_{\hat{r}}\cos\theta - A_{\hat{\theta}}\sin\theta) = -4\pi(J_{\hat{r}}\cos\theta - J_{\hat{\theta}}\sin\theta) \ . \tag{27}$$

By multiplying $\sin\theta$ onto both sides of equation (24) and $\cos\theta$ onto both sides of equation (25), sum up the final

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equations and after performing simple algebraic manipulations, we obtain the following equation:

$$\left(\Delta - \frac{1}{r^2\sin^2\theta}\right)(A_{\hat{r}}\sin\theta + A_{\hat{\theta}}\cos\theta) - \frac{2}{r^2\sin^2\theta}\partial_\phi A_{\hat{\phi}} = -4\pi(J_{\hat{r}}\sin\theta + J_{\hat{\theta}}\cos\theta)\ , \tag{28}$$

The equation (26) can be rewritten as

$$\left(\Delta - \frac{1}{r^2\sin^2\theta}\right)A_{\hat{\phi}} + \frac{2}{r^2\sin^2\theta}\partial_\phi(A_{\hat{r}}\sin\theta + A_{\hat{\theta}}\cos\theta) = -4\pi J_{\hat{\phi}}\ , \tag{29}$$

By multiplying $\cos\phi$ and $\sin\phi$, respectively, onto both sides of equations (28) and (29) (and then (29) and (28)), and after performing simple algebraic manipulations, we obtain

$$\Delta\left[(A_{\hat{r}}\sin\theta + A_{\hat{\theta}}\cos\theta)\cos\phi - A_{\hat{\phi}}\sin\phi\right] = -4\pi\left[(J_{\hat{r}}\sin\theta + J_{\hat{\theta}}\cos\theta)\cos\phi - J_{\hat{\phi}}\sin\phi\right]\ , \tag{30}$$

$$\Delta\left[(A_{\hat{r}}\sin\theta + A_{\hat{\theta}}\cos\theta)\sin\phi + A_{\hat{\phi}}\cos\phi\right] = -4\pi\left[(J_{\hat{r}}\sin\theta + J_{\hat{\theta}}\cos\theta)\sin\phi + J_{\hat{\phi}}\cos\phi\right]\ , \tag{31}$$

which remind the Poisson equations for the components of the vector potential. The solutions of the equations (27), (30) and (31) can be expressed in terms of Green's function as

$$A_{\hat{r}}\cos\theta - A_{\hat{\theta}}\sin\theta = 4\pi\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}(J_{\hat{r}}(\mathbf{r}')\cos\theta' - J_{\hat{\theta}}(\mathbf{r}')\sin\theta')\ ,\ , \tag{32}$$

$$(A_{\hat{r}}\sin\theta + A_{\hat{\theta}}\cos\theta)\cos\phi - A_{\phi}\sin\phi = 4\pi\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\left[(J_{\hat{r}}(\mathbf{r}')\sin\theta' + J_{\hat{\theta}}(\mathbf{r}')\cos\theta')\cos\phi' - J_{\hat{\phi}}(\mathbf{r}')\sin\phi'\right]\ , \tag{33}$$

$$(A_{\hat{r}}\sin\theta + A_{\hat{\theta}}\cos\theta)\sin\phi + A_{\hat{\phi}}\cos\phi = 4\pi\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\left[(J_{\hat{r}}(\mathbf{r}')\sin\theta' + J_{\hat{\theta}}(\mathbf{r}')\cos\theta')\sin\phi' + J_{\hat{\phi}}(\mathbf{r}')\cos\phi'\right]\ . \tag{34}$$

Hereafter performing simple algebraic manipulations the components of vector potential can be found as

$$\begin{aligned} A_{\hat{r}} &= \cos\theta\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}(J_{\hat{r}}(\mathbf{r}')\cos\theta' - J_{\hat{\theta}}(\mathbf{r}')\sin\theta') \\ &\quad + \sin\theta\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\left[(J_{\hat{r}}(\mathbf{r}')\sin\theta' + J_{\hat{\theta}}(\mathbf{r}')\cos\theta')\cos(\phi-\phi') + J_{\hat{\phi}}(\mathbf{r}')\sin(\phi-\phi')\right]\ , \end{aligned} \tag{35}$$

$$\begin{aligned} A_{\hat{\theta}} &= -\sin\theta\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}(J_{\hat{r}}(\mathbf{r}')\cos\theta' - J_{\hat{\theta}}(\mathbf{r}')\sin\theta') \\ &\quad + \cos\theta\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\left[(J_{\hat{r}}(\mathbf{r}')\sin\theta' + J_{\hat{\theta}}(\mathbf{r}')\cos\theta')\cos(\phi-\phi') + J_{\hat{\phi}}(\mathbf{r}')\sin(\phi-\phi')\right]\ , \end{aligned} \tag{36}$$

$$A_{\hat{\phi}} = \int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\left[-(J_{\hat{r}}(\mathbf{r}')\sin\theta' + J_{\hat{\theta}}(\mathbf{r}')\cos\theta')\sin(\phi-\phi') + J_{\hat{\phi}}(\mathbf{r}')\cos(\phi-\phi')\right]\ . \tag{37}$$

The equations (35)-(37) present the solution to the Poisson equations for the components of the vector potential of the electromagnetic fields for an arbitrary given source. Our main target is to find solutions for the vector potential generated by the axially symmetric static current ring.

Now we concentrate on finding the magnetic field generated by a circular static current ring. In this case the current terms are given by $J_{\hat{r}} = J_{\hat{\theta}} = 0$ and $J_{\hat{\phi}} \neq 0$. Then, the the components of the vector potential in (35)-(37) can be rewritten as

$$A_{\hat{r}} = \sin\theta\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}J_{\hat{\phi}}(\mathbf{r}')\sin(\phi-\phi')\ , \tag{38}$$

$$A_{\hat{\theta}} = \cos\theta\int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}J_{\hat{\phi}}(\mathbf{r}')\sin(\phi-\phi')\ , \tag{39}$$

$$A_{\hat{\phi}} = \int\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}J_{\hat{\phi}}(\mathbf{r}')\cos(\phi-\phi')\ . \tag{40}$$

It is worth noting that due to axial symmetry the field should be independent of the angle ϕ. That is why we simply set that $\phi = 0$. Hence, the explicit form of the azimuthal current is given by [12]

$$J_{\hat{\phi}} = \frac{I\sin\theta}{a}\delta(r-a)\delta(\cos\theta)\ , \tag{41}$$

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where I is the total current and a is the radius of the ring. Inserting the expression (41) into (38)-(40), one can obtain the following integrals:

$$\frac{A_{\hat{r}}}{\sin\theta} = \frac{A_{\hat{\theta}}}{\cos\theta} = Ia\int_0^{2\pi} \frac{\sin\phi' d\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}} \ , \tag{42}$$

which is always equal to zero while the azimuthal component of the vector potential takes a form:

$$\begin{aligned} A_{\hat{\phi}} &= Ia\int_0^{2\pi} \frac{\cos\phi' d\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}} \\ &= Ia\left[\frac{r^2+a^2}{2ra\sin\theta}\int_0^{2\pi}\frac{d\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}} - \frac{1}{2ra\sin\theta}\int_0^{2\pi} d\phi'\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}\right] . \end{aligned}$$

The explicit form of the first integral in (43) takes the form:

$$\int_0^{2\pi}\frac{d\phi'}{\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'}} = \frac{2K\left(k_+^2\right)}{r_+} + \frac{2K\left(k_-^2\right)}{r_-} \ , \tag{43}$$

while after introducing $\phi' = 2\chi$, the second term of (43) reads

$$\int_0^{2\pi} d\phi'\sqrt{r^2+a^2-2ra\sin\theta\cos\phi'} = 2\left[r_+E\left(k_+^2\right) + 2r_-E\left(k_-^2\right)\right] \ . \tag{44}$$

Finally, the azimuthal vector potential can be expressed as

$$A_{\hat{\phi}}(r,\theta) = \frac{I}{a}\left\{\frac{r^2+a^2}{ra\sin\theta}\left[\frac{K\left(k_+^2\right)}{r_+} + \frac{K\left(k_-^2\right)}{r_-}\right] + 4k_+^2E\left(k_+^2\right) + 4k_-^2E\left(k_-^2\right)\right\} \ . \tag{45}$$

The non-zero components of the magnetic field can be easily determined as

$$B_{\hat{r}} = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta A_{\hat{\phi}}\right) \ , \qquad B_{\hat{\theta}} = -\frac{1}{r}\frac{\partial}{\partial r}\left(rA_{\hat{\phi}}\right) \ . \tag{46}$$

Figure 1 illustrates the magnetic field line around the static current loop.

V. MAXWELL EQUATIONS

It is also interesting to consider the problem in different ways. One of the promising method is to use the pairs of Maxwell equations. Now, we investigate the same problem directly solving Maxwell equations for the electric field. In the spherical coordinates, the pairs of the Maxwell equations can be expressed as

$$\frac{1}{r^2}\partial_r\left(r^2E_{\hat{r}}\right) + \frac{1}{r\sin\theta}\left[\partial_\theta\left(\sin\theta E_{\hat{\theta}}\right) + \partial_\phi E_{\hat{\phi}}\right] = 4\pi\rho \ , \tag{47}$$

$$\partial_t E_{\hat{r}} = \frac{1}{r\sin\theta}\left[\partial_\theta\left(\sin\theta B_{\hat{\phi}}\right) - \partial_\phi B_{\hat{\theta}}\right] - 4\pi J_{\hat{r}} \ , \tag{48}$$

$$\partial_t E_{\hat{\theta}} = \frac{1}{r\sin\theta}\partial_\phi B_{\hat{r}} - \frac{1}{r}\partial_r\left(rB_{\hat{\phi}}\right) - 4\pi J_{\hat{\theta}} \ , \tag{49}$$

$$\partial_t E_{\hat{\phi}} = \frac{1}{r}\partial_r\left(rB_{\hat{\theta}}\right) - \frac{1}{r}\partial_\theta B_{\hat{r}} - 4\pi J_{\hat{\phi}} \ , \tag{50}$$

and

$$\frac{1}{r^2}\partial_r\left(r^2B_{\hat{r}}\right) + \frac{1}{r\sin\theta}\left[\partial_\theta\left(\sin\theta B_{\hat{\theta}}\right) + \partial_\phi B_{\hat{\phi}}\right] = 0 \ , \tag{51}$$

$$\partial_t B_{\hat{r}} = \frac{1}{r\sin\theta}\left[\partial_\phi E_{\hat{\theta}} - \partial_\theta\left(\sin\theta E_{\hat{\phi}}\right)\right] \ , \tag{52}$$

$$\partial_t B_{\hat{\theta}} = \frac{1}{r}\partial_r\left(rE_{\hat{\phi}}\right) - \frac{1}{r\sin\theta}\partial_\phi E_{\hat{r}} \ , \tag{53}$$

$$\partial_t B_{\hat{\phi}} = \frac{1}{r}\partial_\theta E_{\hat{r}} - \frac{1}{r}\partial_r\left(rE_{\hat{\theta}}\right) \ , \tag{54}$$

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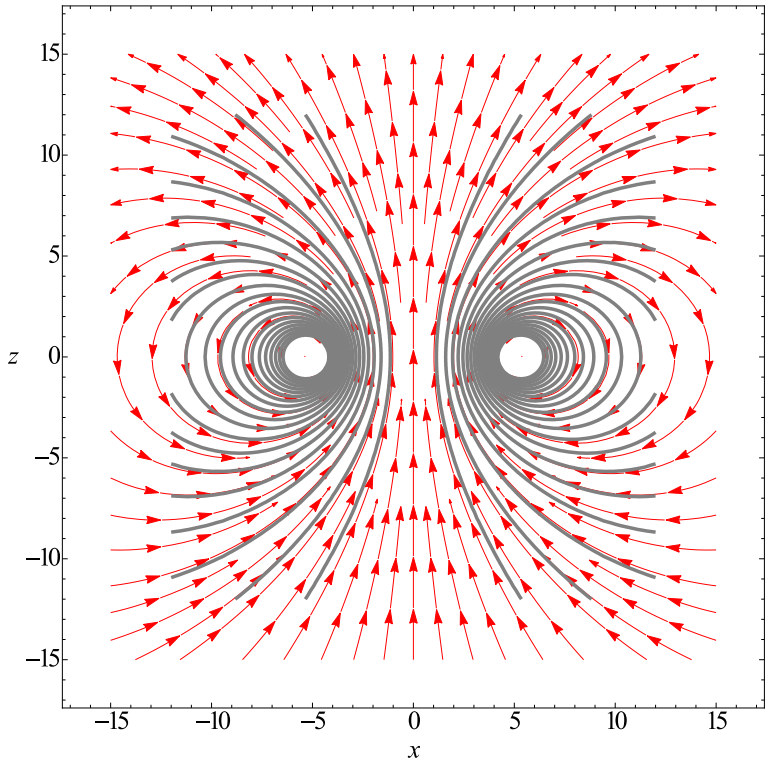


FIG. 2: Magnetic field lines generated by the static current ring in (x, z)-plane.

where $\mathbf{E} = (E_{\hat{r}}, E_{\hat{\theta}}, E_{\hat{\phi}})$ and $\mathbf{B} = (B_{\hat{r}}, B_{\hat{\theta}}, B_{\hat{\phi}})$ are the electric and magnetic fields, ρ and $\mathbf{J} = (J_{\hat{r}}, J_{\hat{\theta}}, J_{\hat{\phi}})$ are the charge and current densities, respectively.

Hereafter taking the time derivative from equation (48) and taking into account equations (53), (54), and (47), it is easy to show that

$$(\Delta - \partial_t^2)(rE_{\hat{r}}) = \frac{4\pi}{r}\partial_r(r^2\rho) + 4\pi r\partial_t J_{\hat{r}} \ , \tag{55}$$

which reminds D'Alembert equation for $rE_{\hat{r}}$ field (See C, for details). However, we are interested in stationary solution of the equation (55), since the system of the charged ring is static the second order time derivative from the electric field should vanish, there is no radial current, i.e. $J_{\hat{r}} = 0$, so that equation (55) reduces to Poisson-like equation as

$$\Delta(rE_{\hat{r}}) = \frac{4\pi}{r}\partial_r(r^2\rho) \ , \tag{56}$$

which is similar to equation (5), but only difference is that instead of the electrostatic potential there stands $rE_{\hat{r}}$, and also the source term is different from that is given in equation (5). Anyway, the solution of the equation (56) can be found as

$$rE_{\hat{r}} = -\frac{q}{2\pi a^2}\int_0^{2\pi} d\phi' \int_0^{\pi} d\theta' \int_0^{\infty} dr' I \ , \tag{57}$$

where

$$I = \frac{r'\partial_{r'}\left[r'^2\delta(r'-a)\right]\sin\theta'\delta(\cos\theta')}{\sqrt{r^2 + r'^2 - 2rr'\left(\cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\phi'\right)}} \ .$$

This integral is similar to (10), however it contains the radial derivative from the delta function. That is why we prefer to change order of integration. It is easy to see that integration over θ is rather simple, and it will be the same as we did before in (10). As we know that integration over angle ϕ' gives the elliptic integral of the first kind. Finally,

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one can obtain

$$E_{\hat{r}} = -\frac{q}{\pi r a^2}\int_0^\infty r' dr' \partial_{r'}\left[r'^2\delta(r'-a)\right]\left[\frac{K\left(\frac{4rr'\sin\theta}{r^2+r'^2+2rr'\sin\theta}\right)}{\sqrt{r^2+r'^2+2rr'\sin\theta}} + \frac{K\left(\frac{-4rr'\sin\theta}{r^2+r'^2-2rr'\sin\theta}\right)}{\sqrt{r^2+r'^2-2rr'\sin\theta}}\right] . \tag{58}$$

Using a property of the delta function $\int f(x)\delta'(x-x_0)dx = -f'(x_0)$, and after performing simple algebra, we obtain the same as expression for the radial electric field as we presented in (14). The tangential component of the electric field can be found from equation (54) for $B_{\hat{\phi}} = 0$, which again agrees with the result previously obtained in (15). In this section, we explicitly show that the solution of Maxwell equations for the components of the electric field due to a uniformly charged ring is exactly the same as that is presented in the previous section.

Hereafter taking the time derivative from equation (52) and taking into account equations (49), (50), and (51), it is easy to show that

$$\left(\Delta - \partial_t^2\right)(rB_{\hat{r}}) = \frac{4\pi}{\sin\theta}\left[\partial_\phi J_{\hat{\theta}} - \partial_\theta(\sin\theta J_{\hat{\phi}})\right] . \tag{59}$$

As we already mentioned that the current ring is static and taking into account equation (41) , above equation takes the form:

$$\Delta\left(rB_{\hat{r}}\right) = -\frac{4\pi}{\sin\theta}\partial_\theta(\sin\theta J_{\hat{\phi}}) . \tag{60}$$

The taking into account the source term (41), the solution of above equation reads

$$\begin{aligned} B_{\hat{r}} &= \frac{I}{ra}\int \frac{r'^2\delta(r'-a)\partial_{\theta'}(\sin^2\theta'\delta(\cos\theta'))dr'd\theta'd\phi'}{\sqrt{r^2+r'^2-2rr'\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi'\right]}} \\ &= \frac{Ia}{r}\int \frac{\partial_{\theta'}(\sin^2\theta'\delta(\cos\theta'))d\theta'd\phi'}{\sqrt{r^2+r'^2-2rr'\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos\phi'\right]}} . \end{aligned} \tag{61}$$

VI. DISCUSSION

According to the standard text books in classical electrodynamics course the expression for the electrostatic potential due to a uniformly charged ring is known as in the series form (See, for example [12, 13])

$$\Phi(r,\theta) = q\sum_\ell^\infty P_\ell(0) r_<^\ell\left(\frac{1}{r_>^{\ell+1}} - \frac{r_>^{\ell+1}}{a^{2\ell+1}}\right)P_\ell(\cos\theta) , \tag{62}$$

where $P_\ell(x)$ is the Legendre polynomial. The first problem of this solution is that we always need to be careful when consider the potential in inside or outside the ring since the solution is given by sum of the interior and exterior solutions. The second problem is related to accuracy of the solution.

It is also interesting to investigate the gravitational analogue of the potential (13), or the gravitational potential due to the ring of the mass m. The structure of the solution for the gravitational potential will be exactly the same as (13), however instead of the electric charge q, the mass of the ring m should stand as

$$\Phi = -\frac{Gm}{\pi}\left[\frac{K\left(\frac{4ra\sin\theta}{r^2+a^2+2ra\sin\theta}\right)}{\sqrt{r^2+a^2+2ra\sin\theta}} + \frac{K\left(\frac{-4ra\sin\theta}{r^2+a^2-2ra\sin\theta}\right)}{\sqrt{r^2+a^2-2ra\sin\theta}}\right] , \tag{63}$$

where G is the gravitational constant. It must be pointed out that the nature of the gravitational interaction is attractive unlike the electromagnetic interaction that is why the negative sign "-" has been included in the final expression. Similarly, the expression for the gravitational acceleration can be derived as written in (14) and (15), but again one has to replace the electric charge q with the quantity $-Gm$. One of the big differences between the electromagnetic and gravitational fields is that the superposition principle works pretty well for the electromagnetic field, however, for the gravitational field it does not work. The another important thing is that in the center of the massive ring gravitational acceleration is zero, that is why it is common called the Lagrange point which is a quite common phenomenon in astronomy, astrophysics, and planetary science.

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VII. CONCLUSIONS

In the present note, we investigate the electrostatic potential and electric field due to the axially-symmetric uniformly charged ring at any point of the space. Similarly, we study the magnetic field due to the axially-symmetric, static current ring at any point of the space. It has been shown that the exact expressions for the electric field due to charged ring and magnetic field due to current ring can be obtained by using two different methods, namely, Poisson and Maxwell equations. The obtained results are concluded as follows:

(i) In the spherical coordinates, the Poisson equation has been solved for the charged uniformly distributed along axially-symmetric ring. The thickness of the ring to be taken negligibly small that than its size.

(ii) By using the property of the delta function, it is shown that 3D integrals in spherical coordinates can be reduced to 1D integral. Finally, the solution for the electrostatic potential can be described by well known the complete elliptic integral of the kind $K(k)$, while the components of the electric field can be expressed in terms of both elliptic integrals of the first and second kind as well. The obtained results coincide with thus previously obtained results in center, at the symmetry axe and the ring's plane.

(iii) We have shown that it is possible to derive the Poission-like equation for only radial component of the electric field, by directly using pairs of Maxwell equations in the spherical coordinates, however it is shown that it is impossible to get the second order separated equations for other components of the electric field. The solution of the equation for radial electric field coincides with the result obtained previously, and it is expressed in terms of the elliptic functions of the first and second kind.

(iv) Our analyses show that the expressions for the electrostatic potential and the electric field can also be applicable for the gravitational potential and the gravitational acceleration due to massive ring, but only thing is the mass of the ring should stand instead of the electric charge with negative sign in the expressions, (i.e. $q \to -Gm$).

(iv) In comparison with previously obtained analytical form of the electrostatic potential and the electric field, our results are much simple and easily applicable in the different areas of the physics related to electromagnetism.

(v) In the spherical coordinates, the Poisson equation for the components of the vector potential of the electromagnetic fields has been has explicitly derived. It is shown that obtained Poisson equations is not for single component of the vector potential but the combination of the component of the vector potential. The general solutions of Poisson equation for the single components of the vector potential depends on all three components of the source terms. As a result, one of the particular solution for the vector potential generated by the axially symmetric, static current ring has been found. It is assumed that the thickness of the ring to be taken negligibly small that than size of the ring.

(vi) By using the property of the delta function, it is shown that 3D integrals in spherical coordinates can be reduced to 1D integral. Finally, the solution for the azimuthal vector potential and the component of the magnetic field can be expressed in terms of the well known the complete elliptic integral of the first and second kind ($K(k)$ and $E(k)$). The obtained results coincide with thus previously obtained results in center, at the symmetry axe and the ring's plane.

(vii) it is also shown possible derivation the Poission-like equation for only radial magnetic field, considering of Maxwell equations in the spherical coordinates. The solution of the equation for radial magnetic field coincides with the result obtained previously, and it is expressed in terms of the elliptic functions of the first and second kind.

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Appendix A: The coordinate transformations

The Cartesian coordinates (x, y, z) are connected to spherical coordinates (r, θ, ϕ) in the usual way:

$$x = r\cos\phi\sin\theta\ , \quad y = r\sin\phi\sin\theta\ , \quad z = r\cos\theta\ , \tag{A1}$$

$$r = \sqrt{x^2+y^2+z^2}\ , \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\ , \quad \phi = \tan^{-1}\frac{y}{x}\ , \tag{A2}$$

where $0 \le r \le \infty$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

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The transformation of the volume element from Cartesian to spherical coordinates is given by the following expression: $d\mathbf{r} = dxdydz = J\, drd\theta d\phi$, where J is the Jacobian determined as

$$J = \begin{pmatrix} \partial_r x & \partial_r y & \partial_r z \\ \partial_\theta x & \partial_\theta y & \partial_\theta z \\ \partial_\phi x & \partial_\phi y & \partial_\phi z \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta \\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta \\ -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0 \end{pmatrix} = r^2\sin\theta\ .$$

The transformation of the Green's function from Cartesian to spherical coordinates can be done by using the following calculations:

$$\begin{aligned} |\mathbf{r}-\mathbf{r}'| &= \sqrt{(x-x')^2+(y-y')^2+(z-z')^2} \\ &= \sqrt{r^2+r'^2-2rr'\left[\cos\theta\cos\theta'+\sin\theta\sin\theta'\cos(\phi-\phi')\right]}\ , \end{aligned} \tag{A3}$$

Appendix B: Elliptic integrals

The explicit calculations of the expression (12) in the form of the elliptic integral of the first kind:

$$\begin{aligned} \Phi(r,\theta) &= \frac{q}{\pi}\int_0^{\pi}\frac{d\chi}{\sqrt{r^2+a^2-2ra\sin\theta+4ra\sin\theta\sin^2\chi}} \\ &= \frac{q}{\pi}\int_0^{\pi/2}\frac{d\chi}{\sqrt{r^2+a^2-2ra\sin\theta+4ra\sin\theta\sin^2\chi}} + \frac{q}{\pi}\int_{\pi/2}^{\pi}\frac{d\chi}{\sqrt{r^2+a^2-2ra\sin\theta+4ra\sin\theta\sin^2\chi}} \\ &= \frac{q}{\pi}\int_0^{\pi/2}\frac{d\chi}{\sqrt{r^2+a^2-2ra\sin\theta+4ra\sin\theta\sin^2\chi}} + \frac{q}{\pi}\int_0^{\pi/2}\frac{d\chi}{\sqrt{r^2+a^2-2ra\sin\theta+4ra\sin\theta\cos^2\chi}} \\ &= \frac{q}{\pi}\int_0^{\pi/2}\frac{d\chi}{\sqrt{r^2+a^2-2ra\sin\theta+4ra\sin\theta\sin^2\chi}} + \frac{q}{\pi}\int_0^{\pi/2}\frac{d\chi}{\sqrt{r^2+a^2+2ra\sin\theta-4ra\sin\theta\sin^2\chi}} \\ &= \frac{q}{\pi r_-}\int_0^{\pi/2}\frac{d\chi}{\sqrt{1+\frac{4ra\sin\theta}{r^2+a^2-2ra\sin\theta}\sin^2\chi}} + \frac{q}{\pi r_+}\int_0^{\pi/2}\frac{d\chi}{\sqrt{1-\frac{4ra\sin\theta}{r^2+a^2+2ra\sin\theta}\sin^2\chi}}\ . \end{aligned} \tag{B1}$$

Finally, we have

$$\begin{aligned} \Phi = &\frac{q}{\pi\sqrt{r^2+a^2-2ra\sin\theta}}K\left(\frac{-4ra\sin\theta}{r^2+a^2-2ra\sin\theta}\right) \\ &+\frac{q}{\pi\sqrt{r^2+a^2+2ra\sin\theta}}K\left(\frac{4ra\sin\theta}{r^2+a^2+2ra\sin\theta}\right)\ . \end{aligned} \tag{B2}$$

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Very similar integral can be evaluated as

$$\begin{aligned}
&\int_0^{2\pi} d\phi' \sqrt{r^2+a^2-2ra\sin\theta\cos\phi'} \\
&= 2\int_0^{\pi} d\chi \sqrt{r^2+a^2-2ra\sin\theta\cos 2\chi} \\
&= 2\int_0^{\pi/2} d\chi \sqrt{r^2+a^2-2ra\sin\theta\cos 2\chi} \\
&+ 2\int_{\pi/2}^{\pi} d\chi \sqrt{r^2+a^2-2ra\sin\theta\cos 2\chi} \\
&= 2\int_0^{\pi/2} d\chi \sqrt{r^2+a^2-2ra\sin\theta+4ra\sin\theta\sin^2\chi} \\
&+ 2\int_0^{\pi/2} d\chi \sqrt{r^2+a^2+2ra\sin\theta-4ra\sin\theta\sin^2\chi} \\
&= 2\sqrt{r^2+a^2-2ra\sin\theta}E\left(\frac{-4ra\sin\theta}{r^2+a^2-2ra\sin\theta}\right) \\
&+ 2\sqrt{r^2+a^2+2ra\sin\theta}E\left(\frac{4ra\sin\theta}{r^2+a^2+2ra\sin\theta}\right) \ .
\end{aligned} \tag{B3}$$

Appendix C: Maxwell equations

The explicit form of Maxwell equations in the spherical coordinates are given by

$$\frac{1}{r^2}\partial_r\left(r^2E_{\hat{r}}\right)+\frac{1}{r\sin\theta}\left[\partial_\theta\left(\sin\theta E_{\hat{\theta}}\right)+\partial_\phi E_{\hat{\phi}}\right]=4\pi\rho \ , \tag{C1}$$

$$\partial_t E_{\hat{r}}=\frac{1}{r\sin\theta}\left[\partial_\theta\left(\sin\theta B_{\hat{\phi}}\right)-\partial_\phi B_{\hat{\theta}}\right]-4\pi J_{\hat{r}} \ , \tag{C2}$$

$$\partial_t E_{\hat{\theta}}=\frac{1}{r\sin\theta}\partial_\phi B_{\hat{r}}-\frac{1}{r}\partial_r\left(rB_{\hat{\phi}}\right)-4\pi J_{\hat{\theta}} \ , \tag{C3}$$

$$\partial_t E_{\hat{\phi}}=\frac{1}{r}\partial_r\left(rB_{\hat{\theta}}\right)-\frac{1}{r}\partial_\theta B_{\hat{r}}-4\pi J_{\hat{\phi}} \ . \tag{C4}$$

and

$$\frac{1}{r^2}\partial_r\left(r^2B_{\hat{r}}\right)+\frac{1}{r\sin\theta}\left[\partial_\theta\left(\sin\theta B_{\hat{\theta}}\right)+\partial_\phi B_{\hat{\phi}}\right]=0 \ , \tag{C5}$$

$$\partial_t B_{\hat{r}}=\frac{1}{r\sin\theta}\left[\partial_\phi E_{\hat{\theta}}-\partial_\theta\left(\sin\theta E_{\hat{\phi}}\right)\right] \ , \tag{C6}$$

$$\partial_t B_{\hat{\theta}}=\frac{1}{r}\partial_r\left(rE_{\hat{\phi}}\right)-\frac{1}{r\sin\theta}\partial_\phi E_{\hat{r}} \ , \tag{C7}$$

$$\partial_t B_{\hat{\phi}}=\frac{1}{r}\partial_\theta E_{\hat{r}}-\frac{1}{r}\partial_r\left(rE_{\hat{\theta}}\right) \ . \tag{C8}$$

We first take time derivative from both sides of the equation (C2) and obtain

$$\partial_t^2 E_{\hat{r}}=\frac{1}{r\sin\theta}\left[\partial_\theta\left(\sin\theta\partial_t B_{\hat{\phi}}\right)-\partial_\phi\partial_t B_{\hat{\theta}}\right]-4\pi\partial_t J_{\hat{r}} \ . \tag{C9}$$

Now, using the expressions (C7) and (C8), $\partial_t B_{\hat{\theta}}$ and $\partial_t B_{\hat{\phi}}$ can be eliminated as

$$\partial_t^2 E_{\hat{r}}=\frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta E_{\hat{r}}\right)+\frac{1}{r^2\sin^2\theta}\partial_\phi^2 E_{\hat{r}}-\frac{1}{r^2}\partial_r\left[\frac{r}{\sin\theta}\left(\partial_\theta\left(\sin\theta E_{\hat{\theta}}\right)+\partial_\phi E_{\hat{\phi}}\right)\right]-4\pi\partial_t J_{\hat{r}} \ , \tag{C10}$$

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Finally,taking into account (C1), one can get

$$\begin{aligned}\partial_t^2 E_{\hat{r}} &= \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta E_{\hat{r}}\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 E_{\hat{r}} - \frac{1}{r^2}\partial_r\left[-\partial_r(r^2E_{\hat{r}}) + 4\pi r^2\rho\right] - 4\pi\partial_t J_{\hat{r}} \\ &= \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta E_{\hat{r}}\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 E_{\hat{r}} + \frac{1}{r^2}\partial_r^2(r^2E_{\hat{r}}) - \frac{4\pi}{r^2}\partial_r(r^2\rho) - 4\pi\partial_t J_{\hat{r}}\ . \end{aligned} \tag{C11}$$

By multiplying r to both sides of this equation, one can have

$$(\Delta - \partial_t^2)(rE_{\hat{r}}) = \frac{4\pi}{r}\partial_r(r^2\rho) + 4\pi r\partial_t J_{\hat{r}}\ . \tag{C12}$$

Very similar calculations can be done for the radial component of the magnetic field by taking time derivative from equation (C6), and obtain

$$\begin{aligned}\partial_t^2 B_{\hat{r}} &= \frac{1}{r\sin\theta}\left[\partial_\phi\partial_t E_{\hat{\theta}} - \partial_\theta\left(\sin\theta\partial_t E_{\hat{\phi}}\right)\right] \\ &= \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta B_{\hat{r}}\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 B_{\hat{r}} - \frac{1}{r^2}\partial_r\left[\frac{r}{\sin\theta}\left(\partial_\theta\left(\sin\theta B_{\hat{\theta}}\right) + \partial_\phi B_{\hat{\phi}}\right)\right] - \frac{4\pi}{r\sin\theta}\left[\partial_\phi J_{\hat{\theta}} - \partial_\theta(\sin\theta J_{\hat{\phi}})\right] \\ &= \frac{1}{r^2}\partial_r(r^2B_{\hat{r}}) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta B_{\hat{r}}\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 B_{\hat{r}} - \frac{4\pi}{r\sin\theta}\left[\partial_\phi J_{\hat{\theta}} - \partial_\theta(\sin\theta J_{\hat{\phi}})\right]\ , \end{aligned} \tag{C13}$$

and

$$\left(\Delta - \partial_t^2\right)(rB_{\hat{r}}) = \frac{4\pi}{\sin\theta}\left[\partial_\phi J_{\hat{\theta}} - \partial_\theta(\sin\theta J_{\hat{\phi}})\right]\ . \tag{C14}$$

Similarly, the second order equations can be obtained for other components of the electromagnetic field as

$$\partial_t^2 E_{\hat{\theta}} = \frac{1}{r}\partial_r^2(rE_{\hat{\theta}}) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 E_{\hat{\theta}} - \frac{1}{r}\partial_\theta\partial_r E_{\hat{r}} - \frac{1}{r^2\sin^2\theta}\partial_\theta\left(\sin\theta\partial_\phi E_{\hat{\phi}}\right) - 4\pi\partial_t J_{\hat{\theta}}\ , \tag{C15}$$

$$\partial_t^2 E_{\hat{\phi}} = \frac{1}{r}\partial_r^2(rE_{\hat{\phi}}) + \frac{1}{r^2}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta E_{\hat{\phi}})\right] - \frac{1}{r\sin\theta}\partial_\phi\partial_r E_{\hat{r}} - \frac{1}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\phi E_{\hat{\theta}}\right) - 4\pi\partial_t J_{\hat{\phi}}\ , \tag{C16}$$

$$\partial_t^2 B_{\hat{\theta}} = \frac{1}{r}\partial_r^2(rB_{\hat{\theta}}) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2 B_{\hat{\theta}} - \frac{1}{r}\partial_\theta\partial_r B_{\hat{r}} - \frac{1}{r^2\sin^2\theta}\partial_\theta\left(\sin\theta\partial_\phi B_{\hat{\phi}}\right) + \frac{4\pi}{r\sin\theta}\partial_\phi J_{\hat{r}} - \frac{4\pi}{r}\partial_r(rJ_{\hat{\phi}})\ , \tag{C17}$$

$$\partial_t^2 B_{\hat{\phi}} = \frac{1}{r}\partial_r^2(rB_{\hat{\phi}}) + \frac{1}{r^2}\partial_\theta\left[\frac{1}{\sin\theta}\partial_\theta(\sin\theta B_{\hat{\phi}})\right] - \frac{1}{r\sin\theta}\partial_\phi\partial_r B_{\hat{r}} - \frac{1}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\phi B_{\hat{\theta}}\right) - \frac{4\pi}{r}\partial_r(rJ_{\hat{\theta}}) - \frac{4\pi}{r}\partial_\theta J_r\ . \tag{C18}$$

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