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# Absorption of particles by global monopole

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We study motion of massless (photon) and massive test particles around the Schwarzschild black hole with global monopole. It is shown that a global monopole strengthens the gravitational attraction of black hole as the radius of event horizon increases, and consequently, radii of all the characteristic circular orbits, such as radius of photonsphere, marginally bound and marginally stable circular orbits increase on account of the global monopole. Furthermore, we have studied the absorption cross sections of massless (photon) and massive particles by the Schwarzschild black hole with global monopole and shown that they also increase with increasing the value of global monopole. Moreover, we have calculated the capture cross section of test relativistic and non-relativistic particles and obtained very interesting result that in the non-relativistic regime ($E \to 1$), the critical impact parameter and capture cross section tends to finite value, unlike the case of the Schwarzschild black hole.

## I. INTRODUCTION

A black hole is one of the most fascinating object whose existence was predicted by general theory of relativity a century ago, as it can be described by the only three parameters: mass, angular momentum and charge, according to the no-hair theorem. Its existence has recently been proven thanks to the technological development by the detected gravitational wave signals from coalescence of black holes/neutron stars in binary [1–6] by the LIGO-Virgo collaborations and by the observational evidence based on electromagnetic radiation coming from accretion disks around astrophysical black holes [7] as well as from the detected first ever image of the supermassive black hole at the center of the galaxy M87 by the Event Horizon Telescope (EHT) collaboration [8, 9]. Despite these breakthrough discoveries confirmed the existence of the black hole and correctness of the general relativity in the weak gravitational regime [10], its correctness in the strong field regime is still under question. This is leaving a door wide open for the alternative theories of gravity [11]. The alternative theories of gravity are constructed by either considering higher orders of the curvature of the spacetime such as $f(r)$-gravity, Gauss-Bonnet gravity, Lovelock gravity, etc., or coupling the general relativity or higher order curvatures to the other fundamental fields, such as electromagnetic, scalar, dirac, etc. fields. One of such theories was proposed by Barriola and Vilenkin that is a global monopole that is obtained by the coupling general relativity and a self-coupling triplet of scalar fields, so-called Goldstone fields with energy density decreasing with the distance as $r^{-2}$ [12]. According to the theory, the current global monopole is a topological defects being a result of a global $O(3)$ symmetry breaking which may have been arosen during the phase transitions in the early universe. Afterwards, various physical phenomena around global monopole and it is related black holes [13–23]. In this paper, we aim to study not only orbits around, but also the capture of massive and massless particles by the global monopole. The paper is organized as follows. The next section II is devoted to the description of Schwarzschild black hole with global monopole solution and its main properties. In section III we present equations of motion of photon and massive particle in the equatorial plane of the black hole. In section IV a motion of photon and its capture by the Schwarzschild black hole with global monopole is studied. In the section V we study the capture cross section of massive test particle by the Schwarzschild black hole with global monopole. Finally, in the last section VI we summarize main results obtained in the paper. Throughout the paper we use the geometerized units in which the Newtonian gravitational constant $G$, speed of light $c$ are set as $G = c = 1$.

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## II. BASIC EQUATIONS

The general relativity coupled to the scalar field generating global monopole is described by the action

$$\int d^4x\sqrt{-g}\,(R+\mathcal{L}_{GM})\ , \tag{1}$$

where the Lagrangian density of global monopole (GM) is given by

$$\mathcal{L}_{GM}=\frac{1}{2}\partial_\mu\phi^a\partial^\mu\phi^a-\frac{\lambda}{4}(\phi^a\phi^a-\eta^2)^2\ , \tag{2}$$

where the scalar field is given by

$$\phi^a=\eta f(r)\frac{x^a}{r}\ ,\quad a=1,2,3\ , \tag{3}$$

where $x^a$ is Cartesian coordinates ($x^ax^a=r^2$), $\lambda$ is the self-interaction term. As one noticed without loss of generality we adopted units such that the gravitational constant and speed of the light are equal to unit i.e. $G=c=1$. After applying the least action principle to the action (1) we obtain Einstein field equations as

$$R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R=8\pi T_{\mu\nu}\ , \tag{4}$$

where $R_{\mu\nu}$, $R$, and $g_{\mu\nu}$ are Ricci tensor, Ricci scalar and metric tensor, respectively. The energy-momentum tensor of the global monopole is given by

$$\begin{aligned}
T_t^t&=-\eta^2\left(\frac{f^2}{r^2}+\frac{N^2f'^2}{2}\right)-\frac{\eta^4\lambda}{4}\left(f^2-1\right)^2\ ,\\
T_r^r&=-\eta^2\left(\frac{f^2}{r^2}-\frac{N^2f'^2}{2}\right)-\frac{\eta^4\lambda}{4}\left(f^2-1\right)^2\ ,\\
T_\theta^\theta&=-\frac{\eta^2N^2f'^2}{2}-\frac{\eta^4\lambda}{4}\left(f^2-1\right)^2=T_\varphi^\varphi.
\end{aligned} \tag{5}$$

By solving the Einstein field equations (4) for the spacetime with topology $\mathcal{R}\times\mathcal{S}^2$ one can obtain the following Schwarzschild black hole solution with global monopole [12]:

$$ds^2=-N^2(r)dt^2+\frac{dr^2}{N^2(r)}+r^2d\Omega^2, \tag{6}$$

where

$$N^2(r)=1-\eta^2-\frac{2M}{r}\ , \tag{7}$$

and the line element of two dimensional sphere is given by

$$d\Omega^2=d\theta^2+\sin^2\theta d\varphi^2, \tag{8}$$

We must here note that this solution is asymptotically non-flat. The event horizon of the spacetime is given by the divergence of the spacetime metric through $g_{rr}$ that gives

$$r_h=\frac{2M}{1-\eta^2}\ . \tag{9}$$

From expression (9) one can see that when the parameter $\eta$ vanishes, one recovers the event horizon of the Schwarzschild black hole, while when $\eta$ tends to unity from the left side, the event horizon tends to infinity. For the values of the global monopole $\eta\geq 1$, the time and radial components of the metric tensor becomes space-like and time-like, respectively, that is beyond our considerations.


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## III. CIRCULAR ORBITS

In this section we study the test particle's motion in the spacetime of the global monopole and assume that the motion in confined at the equatorial plane of the spacetime so as $\theta = \pi/2$. Then, one can easily notice from the symmetry of the spacetime metric (6) that the momenta corresponding to the time, $t$, and azimuthal coordinate, $\varphi$, are conserved due to the stationarity and spherical symmetry of the spacetime, and these conserved momenta are called energy, $E$, and angular momentum, $L$, of the particle, respectively, as

$$\left(1-\eta^2-\frac{2M}{r}\right)u^t = E\,,$$
$$r^2 u^\varphi = L\,. \tag{10}$$

From the normalization condition $u_\mu u^\mu = -\epsilon$, we obtain the following equation for the radial velocity of the particle:

$$(u^r)^2 = E^2 - V_{\rm eff} \quad \text{with} \quad V_{\rm eff} = N^2(r)\left(\frac{L^2}{r^2}+\epsilon\right), \tag{11}$$

where $\epsilon$ is a mass of the test particle and it is either $\epsilon = 1$ or $\epsilon = 0$ for the massive and massless (photon) particles, respectively. One can see from Fig. 1 that with increasing the dimension of the spacetime, height of the effective potential increases, however, the maximum of the effective potential which corresponds to the unstable circular photon orbit shifts towards the central black hole.

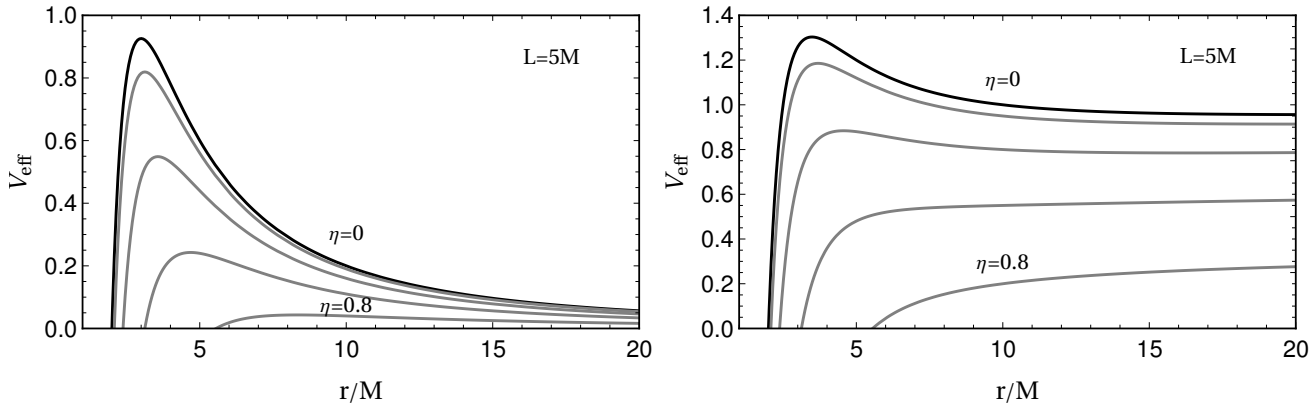



FIG. 1: Radial profile of the effective potential of photon (left panel) and massive particle (right panel) in the field of the Schwarzschild black hole with global monopole for $\mathrm{L} = 5\mathrm{M}$ and the parameter is in the range $\eta \in [0, 0.8]$.

Now by using the conditions for the particle to revolve around the black hole, it must have no radial velocity and its orbit corresponds to the extrema of the effective potential, respectively, as

$$u^r = 0\,, \qquad V'_{\rm eff} = 0\,, \tag{12}$$

one can find the specific energy and specific angular momentum of the particle moving along the circular orbit around Schwarzschild-Tangherlini black hole as

$$\mathcal{E}^2 = -\frac{\left[\left(\eta^2-1\right)r+2M\right]^2}{r\left[\left(\eta^2-1\right)r+3M\right]}\,, \tag{13}$$

$$\mathcal{L}^2 = -\frac{Mr^2}{\left(\eta^2-1\right)r+3M}\,. \tag{14}$$

Note that the specific energy (or angular momentum) of the particle defines the energy (or angular momentum) corresponding to per mass of the particle as $\mathcal{E} = E/\epsilon$ (or $\mathcal{L} = L/\epsilon$). Taking the fact that the energy and angular momentum of the photon in the circular orbit diverge into consideration, one can easily find the radius of the photonsphere in the Schwarzschild black hole with global monopole black hole spacetime as

$$\mathrm{r_{ps}} = \frac{3M}{1-\eta^2}\,, \tag{15}$$

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In Fig. 2 the radius of the photonsphere for as a function of the golobal monopole of the spacetime is presented. In the case of standard Schwarzschild black hole, one recovers the photonsphere at $\mathrm{r_{ps}} = 3M$. One can see from Eq. (15) and Fig. 1 that a global monopole in the spacetime strengthens the gravitational attraction of the black hole and consequently, the radius of the photonsphere increases.

One more astrophysically relevant circular orbits around the black hole are the marginally stable circular orbits (MSCO). It is obvious that the stability of the orbit is ensured by positivity of the second derivative of effective potential with respect to radial coordinate that provides the effective potential to be concave upward. The marginally stable orbits are located at the following inflection point where the concavity of the effective potential changes:

$$\mathrm{V''_{eff}} = 0\,. \tag{16}$$

To find the MSCOs, one must solve Eq. (17) together with the circularity conditions Eq. (12). By solving them we determine that the MSCO is bounded only from the inner side. Therefore, we call this orbit innermost stable circular orbit (ISCO) and it is located at

$$\mathrm{r_{isco}} = \frac{6M}{1-\eta^2}\,, \tag{17}$$

In the case of standard Schwarzschild black hole limit ($\eta = 0$), one recovers the ISCO radius at $\mathrm{r_{isco}} = 6\mathrm{M}$. With increasing the value of the global monopole, it increases and at $\eta = 1$, it diverges.

The next orbit of the test particle we want to mention here is the marginally bound circular orbit (MBCO). The MBCO represents the innermost circular orbit of a massive particle which is energetically bounded to the central black hole. Surely, this radius is located closer to the black hole than ISCO. Between these two orbits there can still be circular orbits but they all are unstable. Since the spacetime we are considering in this paper is asymptotically non-flat and it tends to $1-\eta^2$ at inifnity, the MBCO is determined as solution of the equation of $\mathcal{E}(\mathrm{r_{mbo}}) = 1-\eta^2$. By solving this equation we find the radius of MBCO to be located at

$$\mathrm{r_{mbo}} = \frac{4M}{1-\eta^2}\,, \tag{18}$$

In the case of standard Schwarzschild black hole limit ($\eta = 0$), one recovers the MBCO radius at $\mathrm{r_{mbo}} = 4\mathrm{M}$. In Fig. 2 we sum all results obtained from the circular orbits. From these results we obtained that for the small values of the global monopole, radii of all characteristic orbits slowly increase from the ones around the Schwarzschild black hole. For the values of the global monopole close to the unity, these radii rapidly increase and at $\eta = 1$, they diverge.

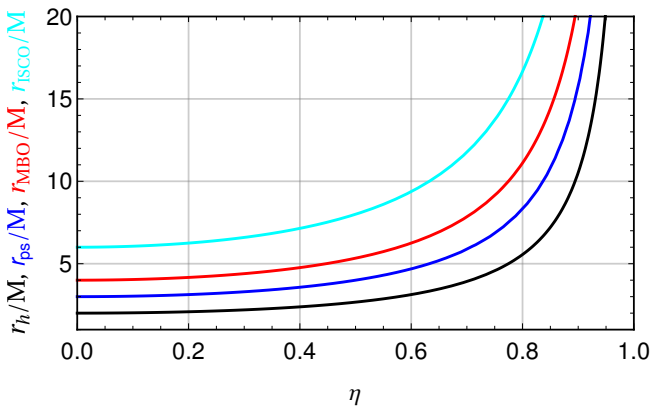



FIG. 2: Radii of the characteristic circular orbits of the test particle in the Schwarzschild global monopole spacetime as a function of the spacetime parameter $\eta$.

## IV. CAPTURE OF PHOTON

In this section we study a motion of photon and its capture by the Schwarzschild black hole with global monopole whose line element is given by (6) with metric function (7). Prior to turning to the study we introduce a notion of impact parameter, denoted by $b$, that defines the possible closest approach distance of photon to the black hole before reaching an observer at infinity. It

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is well-known that the photon passing through the black hole spacetime can approach the black hole until the photonsphere, beyond it move along the photonsphere or is captured by the black hole. Therefore, the photonsphere is a boundary between capture and escape scenarios. Therefore, our main task in this section is to find the impact parameter, $b$, of the photon.

$$b_{\rm cr} = \frac{L_{\rm cr}}{E} = \frac{3\sqrt{3}M}{(1+\eta^2)^{3/2}}, \tag{19}$$

with $L_{\rm cr}$ being the critical angular momentum of photon. For simplicity, one can rewrite the radial equation of motion (11) for the photon ($\epsilon = 0$) with applying the transformation $\lambda \to \lambda L$ for the affine parameter in the following polynomial form:

$$(u^r)^2 = \frac{1}{b^2} - \frac{N^2(r)}{r^2}\,. \tag{20}$$

In order for the right-hand side of Equation (20) to take the monic polynomial form, we multiply both sides of equation by $b^2r^3$ as

$$b^2r^3\,(u^r)^2 = r^3 + b^2(\eta^2-1)r + 2b^2\,. \tag{21}$$

The zeros of the right-hand side of Eq. (21) represent the turning points. A shape of the effective potential indicates that in this case, there are three possible scenarios in terms of the numbers of turning points:

(i) no turning point with $b < b_{\rm cr}$, which characterizes the cases that either the photon coming from infinity falls into a black hole, or if a photon that was emitted from near the horizon escapes to infinity;

(ii) one turning point with $b = b_{\rm cr}$ that characterizes the case that the photon coming from infinity or emitted just outside the event horizon of the black hole is captured by the photonsphere at $r_1 = r_{\rm ps}$;

(iii) two turning points with $b > b_{\rm cr}$ characterize the cases that either the photon coming from infinity reaches the periastron ($r_2$) and again escapes to infinity, or if the photon emitted from near the horizon reaches an apastron ($r_1$) and falls back to the black hole.

Thus, we are interested in the second scenario in which the critical impact parameter corresponds to the maximum of the effective potential where two turning points merge to one. To find the value of $b$ for the only one turning point, we must find the value of $b$ that makes discriminant of polynomial on the right hand side of (21). Here to find the discriminant, we apply a determinant of the Sylvester matrix for the resultant of polynomial. The determinant of the polynomial on the right hand side of (21) is determined by the following Sylvester matrix:

$$\begin{pmatrix} 1 & 0 & b^2(\eta^2-1) & 2b^2 & 0 \\ 0 & 1 & 0 & b^2(\eta^2-1) & 2b^2 \\ 3 & 0 & b^2(\eta^2-1) & 0 & 0 \\ 0 & 3 & 0 & b^2(\eta^2-1) & 0 \\ 0 & 0 & 3 & 0 & b^2(\eta^2-1) \end{pmatrix}. \tag{22}$$

Zero of the determinant of the Sylvester matrix gives the critical impact parameter. It equals the following equation:

$$b^2\left(\eta^2-1\right)^3 + 27 = 0\,, \tag{23}$$

By solving the above equation with respect to impact parameter, we found the critical value of impact parameter that corresponds to the radius of shadow of the Schwarzschild black hole with global monopole as

$$R_{sh} \equiv b_{\rm cr} = \frac{3\sqrt{3}M}{\sqrt{(1-\eta^2)^3}}\,, \tag{24}$$

and the geometrical capture cross section of photon by the Schwarzschild black hole is equal to

$$\sigma_{\rm capt} = \frac{27\pi M^2}{(1-\eta^2)^3}. \tag{25}$$

For $\eta = 0$, one recovers the capture cross section of the photon by the Schwarzschild black hole that equals $\sigma_{\rm capt} = 27\pi M^2$ [24–31].

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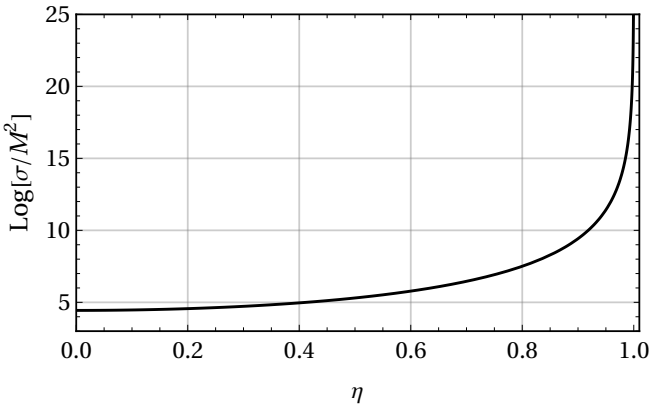



FIG. 3: Radii of the characteristic circular orbits of the test particle in the Schwarzschild global monopole spacetime as a function of the spacetime parameter $\eta$.

## V. CAPTURE OF MASSIVE PARTICLE

In this section we study the capture cross section of massive test particle by the Schwarzschild black hole with global monopole. Similar to the previous section, to study the capture cross section of massive particle by the black hole, one must first determine the critical impact parameter of the particle. However, since the effective potential of the massive particle tends to $1-\eta^2$ at spatial infinity as shown in Fig. 1, not to zero as in the case of photon in Fig. 1, the critical impact parameter is defined quite differently to the case of photon (19), as following:

$$b_{\rm cr} = \frac{\mathcal{L}_{\rm cr}}{\sqrt{\mathcal{E}^2+\eta^2-1}} \,. \tag{26}$$

As it is seen from (26), our main task is to find the critical specific angular momentum of the particle. To find it, we first rewrite radial equation of motion (11) in the following form:

$$\begin{aligned} &r^3(u^r)^2 = \\ &\left(\mathcal{E}^2-1+\eta^2\right) r^3 + 2Mr^2 - \mathcal{L}^2(1-\eta^2)r + 2\mathcal{L}^2 M \,, \end{aligned} \tag{27}$$

The zeros of a polynomial on the right hand side of (27) represent the turning points. As in the case of the capture of photon in the previous section, from a shape of the effective potential in Fig. 1 it is easy to realize that depending on the energy of the particle, there can be up to two turning points and one of the three scenarios presented in previous section can happen. Again the critical impact parameter corresponds to the case with one turning point where two turning points merge. The two turning points merge only if the discriminant of the polynomial on the right hand side of (27) vanishes. Again, to find the discriminant we apply the Sylvester matrix for the resultant of polynomial. The Sylvester matrix for this case is written as

$$\det\begin{pmatrix} \mathcal{E}^2+\eta^2-1 & 2M & \left(\eta^2-1\right)\mathcal{L}^2 & 2\mathcal{L}^2M & 0 \\ 0 & \mathcal{E}^2+\eta^2-1 & 2M & \left(\eta^2-1\right)\mathcal{L}^2 & 2\mathcal{L}^2M \\ 3\left(\mathcal{E}^2+\eta^2-1\right) & 4M & \left(\eta^2-1\right)\mathcal{L}^2 & 0 & 0 \\ 0 & 3\left(\mathcal{E}^2+\eta^2-1\right) & 4M & \left(\eta^2-1\right)\mathcal{L}^2 & 0 \\ 0 & 0 & 3\left(\mathcal{E}^2+\eta^2-1\right) & 4M & \left(\eta^2-1\right)\mathcal{L}^2 \end{pmatrix} = 0 \,, \tag{28}$$

that gives the following equation:

$$\left(\eta^2-1\right)^3\mathcal{L}^4\left(\mathcal{E}^2+\eta^2-1\right) + \mathcal{L}^2M^2\left[27\mathcal{E}^4+36\mathcal{E}^2\left(\eta^2-1\right)+8\left(\eta^2-1\right)^2\right] + 16M^4 = 0, \tag{29}$$

By solving this equation, we obtain the following expression for the specific angular momentum:

$$\mathcal{L}_{\rm cr} = M\sqrt{\frac{27\mathcal{E}^4+36\mathcal{E}^2\left(\eta^2-1\right)+\mathcal{E}\left(9\mathcal{E}^2+8\eta^2-8\right)^{3/2}+8\left(\eta^2-1\right)^2}{2\left(1-\eta^2\right)^3\left(\mathcal{E}^2+\eta^2-1\right)}}, \tag{30}$$


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By inserting expression (30) to (26), one obtains the critical impact parameter and consequently, the geometrical capture cross section of the massive particle with energy per mass $\mathcal{E}$. In Fig. 4 the capture cross section of the massive particle by the Schwarzschild black hole with global monopole as a function of the enery og the particle is given for the different values of the global monopole parameter.

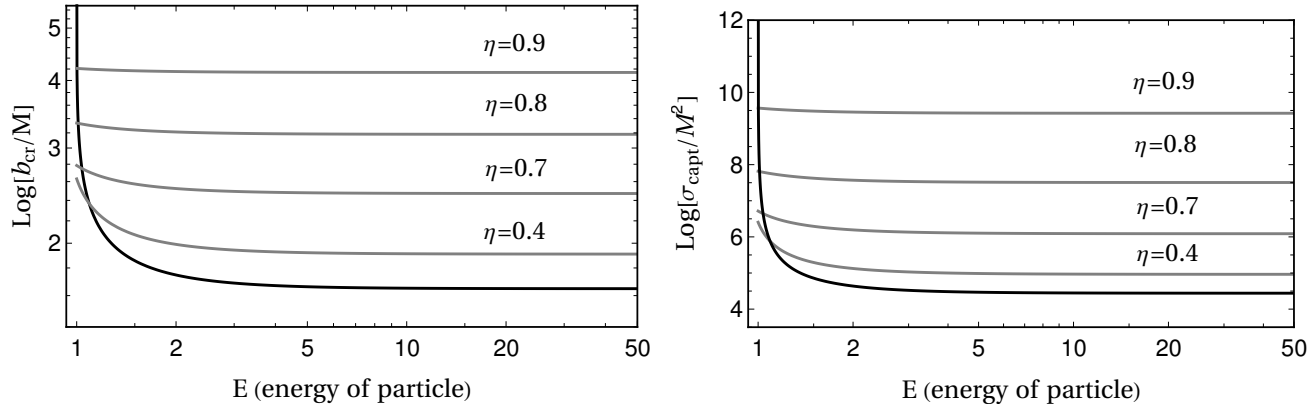



FIG. 4: The critical impact parameter and capture cross section of massive particle by the Schwarzschild black hole with global monopole as a function of the energy of the particle. Where the black line corresponds to the one by the Schwarzschild black hole ($\eta = 0$).

One can see from Fig. 4 that with increasing the value of the global monopole, the probability of the particle to be captured by the Schwarzschild black hole with global monopole increases. Moreover, the capture cross section of the particle rapidly increases for the non-relativistic particles (in low energy regime), while it tends to finite value for the relativistic particles (high energy regime). Let us check these limiting values of the capture cross sections of particle in non-relativistic and relativistic cases below. Let us consider the test particle is relativistic or highly energetic. In this case the capture cross section is given by

$$\sigma_{\rm capt} = \frac{27\pi M^2}{(1-\eta^2)^3}\left(1+\frac{2(1-\eta^2)}{3\mathcal{E}^2}-\frac{17(1-\eta^2)}{27\mathcal{E}^4}\right)+O(\mathcal{E}^{-6}), \tag{31}$$

that in the first order approximation and Schwarzschild limit ($\eta = 0$) becomes identical with the one given in [24].

Now let us consider non-relativistic particle with energy $\mathcal{E} = 1 + \beta^2/2$, where $\beta$ is a ratio of the velocity of the particle and speed of light as $\beta = v/c$. In this case the capture cross section is given by

$$\sigma_{\rm capt} = \frac{\pi M^2\left[\left(1+8\eta^2\right)^{3/2}+8\eta^4+20\eta^2-1\right]}{2\eta^4\left(1-\eta^2\right)^3}+O(\beta^2). \tag{32}$$

It is well known from the pioneering works that the capture cross section of the massive particle at rest ($E = 1$) by the Schwrazschild and other black holes of general relativity diverges. Interestingly, the capture cross section of such particle by the Schwarzschild black hole with global monopole does not diverge, unless $\eta = 1$, instead it tends to finite value. This finite value is given as a function of the global monopole in Fig. 5

## VI. CONCLUSION

In this paper we have studied the characteristic circular orbits of the massive and massless particles around Schwarzschild black hole with global monopole that is a solution of general relativity coupled to the Goldstone field generating global monopole in $\mathcal{R} \times \mathcal{S}^2$ topology. By studying the circular orbits of test particles, we have determined that the global monopole of the black hole strengthens the gravitational attraction of the black hole, consequently, its horizon and radii of all the characteristic circular orbits, such as radii of photonsphere, innermost stable and marginally bound circular orbits increase on account of the global monopole.

Then we have studied the capture cross section of massless (photon) and massive test particles by the Schwarzschild black hole with global monopole. The critical impact parameters and capture cross sections of photon and massive particles are calculated and shown that they also increase with increasing the value of the global monopole. Moreover, we have calculated the capture

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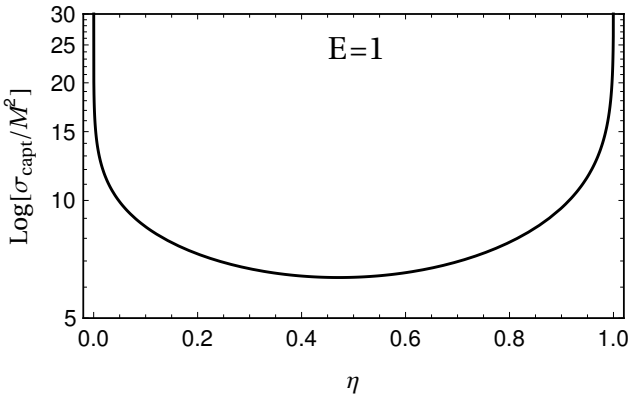



FIG. 5: The capture cross section of massive particle by the Schwarzschild black hole with global monopole as a function of $\eta$ for $E = 1$ case.

cross section of relativistic and non-relativistic test particles. Interestingly, unlike in the case of the Schwarzschild black hole (or other black holes in pure general relativity), the capture cross section (critical impact parameter) of particle at rest by the Schwarzschild black hole with global monopole does not diverge, instead it tends to finite value.

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